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I.—ON THE PROPAGATION OF A WAVE IN AN ELASTIC MEDIUM.

The following investigation of the equation for the propagation of a wave in an elastic medium, may be considered as supplemental to the two articles on the Undulatory Theory in the first and second Numbers of this Journal: though perhaps it ought more properly to have preceded them. But as we do not pretend to offer a complete treatise on the subject, the order of the articles is not a matter of much consequence.

1. Light, according to the wave theory, consists of undulations propagated in an elastic medium. From the laws of the interference of polarized light, it appears that the constituent vibrations are transverse to the direction of propagation; that polarized light consists of vibrations in one direction, or perpendicular to one plane, which in Fresnel's theory is the *plane of polarization*, and in Cauchy's theory is perpendicular to the *plane of polarization;* and thirdly, that there are no vibrations, or at least none capable of producing the sensation of light, in the direction of propagation. Fresnel assigns as the reason for this, that the resistance to compression of the medium is so great, that if a disturbance takes place motion is propagated to a great distance before the disturbed molecule returns to rest. Hence, a wave of great breadth will be propagated, but of feeble intensity, as the *vis viva* remains constant.

2. To this we may add, that the possibility of the propagation of motion to a distance depends upon this, that waves of *constant breadth* can be propagated: if we examine our equations, we see that this depends upon the disturbance being a function of $vt-x$;

and the partial differential equation which gives this form of solution is obtained, in the simplest case, namely that of motion in one direction, by supposing the elastic force proportional to the condensation, and the condensations to be infinitesimal. Without these restrictions the equations cannot be solved; but from several considerations it seems probable, that a particle once disturbed does not immediately come to rest, and that the waves increase in breadth. We may see examples of this in the waves propagated when a stone is thrown into still water. But in water the velocity of a wave depends on its breadth; and thus, though the breadths of the waves increase, they do not interfere with each other.

3. It appears probable, from the change in the character of sounds caused by distance, the loss of sharpness and distinctness, that the breadth of a wave of air increases. If we suppose the same to be the case with those waves of light which consist of vibrations in the direction of propagation, but to a greater degree than in sound, we shall have, in the first place, a rapid diminution in the *vis viva* of each wave; and also, as their velocities are nearly independent of their breadths, each wave will overtake its predecessor and be overtaken by its successor, and there will be an interference which may speedily obliterate all traces of the disturbance.

4. We have now to consider that complex system of disturbances which constitutes an undulation.

We shall suppose the molecules of the media which we consider to be so arranged, that the three rectangular directions, or the three axes of elasticity, have the same direction at every point. These we shall take for coordinate axes, and shall suppose that the elasticity called into play is greatest for a displacement parallel to the axis of x, and least for one parallel to the axis of z, or that the axes of x, y, z are the axes of greatest, mean, and least elasticity.

5. If we suppose a range of particles, in a plane parallel to that of yz, to be moved upwards, or parallel to the axis of z, the resultant of the attraction of all the others will tend to pull it back along the same direction with a force proportional to the displacement: the same thing will happen if it receive a displacement parallel to the axis of y, but for the same displacement the force of restitution will be greater than before, as we have supposed the elasticity for displacements parallel to y to be greater than for those parallel to z. Let a^2, b^2, c^2 be the coefficients of elasticity, so that in the first case the force tending to bring back the range of particles is $c^2 \times$ displacement, and in the other $b^2 \times$ displacement. If the influence of one range of particles does not extend beyond the nearest range, the force of each of the adjacent ranges will pull the displaced range back with a force $= \frac{c^2}{2} \times$ displacement.

Let us now consider a succession of ranges to be displaced by different quantities, all parallel to the axis of z, the ranges being all

at equal intervals. Let the range at a distance x from the origin be displaced by a quantity z_x, that at a distance $x+h$ by a quantity z_{x+h}, and so on. The range at x will be pulled *up* by the range at $x+h$ with a force $\frac{1}{2}c^2(z_{x+h}-z_x)=\frac{1}{2}c^2\Delta z_x$, and *down* by the range at $x-h$ with a force $\frac{1}{2}c^2\Delta z_{x-h}$: thus from these two it will be pulled *up* with a force equal to their difference, or

$$\tfrac{1}{2}c^2(\Delta z_x-\Delta z_{x-h})=\tfrac{1}{2}c^2\Delta^2 z_{x-h};$$

and if M be the mass of the range at x, we have for its motion

$$\frac{d^2z_x}{dt^2}=\frac{c^2}{2\mathrm{M}}\Delta^2 z_{x-h}.$$

6. The general solution of this equation of mixed partial differences has been given in Article II. of our last Number; and there it is shown, that the velocity of propagation depends on the length of the wave, and thus the phenomenon of dispersion is accounted for. In the present case we shall take the approximate solution, where h is very small. By putting $\frac{d^2z_x}{dx^2}h^2$ for $\Delta^2 z_{x-h}$, our equation becomes

$$\frac{d^2z_x}{dt^2}=c^2\frac{h^2}{2\mathrm{M}}\frac{d^2z_x}{dx^2}.$$

This is the well-known equation, which shows that a wave will be propagated along the axis of x with a velocity $c\frac{h}{\sqrt{2\mathrm{M}}}$. If the vibrations take place parallel to the axis of y, there will be a wave propagated with a velocity $b\frac{h}{\sqrt{2\mathrm{M}}}$. Fresnel states, that he has satisfied himself, by experiment, that the velocity of a plane wave depends solely on the direction of the constituent vibrations, and not on the direction of the wave. It would appear from this, that the factors of b and c in the above expressions must be constant for different positions of the wave; and it results, that plane waves, consisting of vibrations parallel to the axes of x, y, or z, will be propagated perpendicularly to themselves with velocities which are proportional to a, b, c, or to the square roots of the coefficients of the elastic force for the disturbance of a single range.

7. If the range of particles in the plane of yz receive a disturbance in that plane, but not parallel to one of the axes, we must resolve this displacement into *two*, one parallel to each axis, and there will result *two* waves, propagated with velocities b and c respectively along the axis of x, and corresponding waves propagated in a negative direction.

And this result follows, whether the displacement takes place along a line, as in polarized light, or in some more complicated manner, as in common light; and thus any kind of disturbance in one of the coordinate planes will produce two waves, polarized at right angles to each other, and of different refrangibility.

8. We have now to consider the general case of a plane wave inclined at any angles to the coordinate planes. A vibration in this plane will produce a force of restitution, which will have a different direction from the displacement. Resolve this force into two, one along the direction of displacement, the other perpendicular to it; if this last portion be also perpendicular to the front of the wave, it may be neglected, and a wave will be propagated with a velocity proportional to the square root of the elasticity resolved along the direction of vibration; if it be not perpendicular to the front of the wave, it cannot be neglected. We have, therefore, first to inquire whether this is the case for any direction of vibration in the plane; and this leads us into the subject which was discussed in our last Number.

A. S.

II.—CIRCULAR SECTIONS IN SURFACES OF THE SECOND ORDER.

In determining the circular sections in the Surface of Elasticity, Fresnel has made use of a method which is very readily applicable to surfaces of the second order; and as it has not yet been introduced into any work on Analytical Geometry, it may be useful to insert it here.

Taking first the surfaces which have a centre, let their equation be

$$Px^2 + P'y^2 + P''z^2 = H \text{ .... } (1),$$

and let this be cut by a plane

$$z = mx + ny \text{ .... } (2),$$

which we suppose to pass through the centre, as all sections made by parallel planes are similar. Let this plane also cut the sphere

$$x^2 + y^2 + z^2 = r^2 \text{ .... } (3).$$

Now, as m, n, r are indeterminate, we can so assume the position of the plane and the magnitude of the sphere, that the circular section of the surface (1), if it exist, shall coincide with the section of the sphere; and if these coincide, the equations to their projections on the plane of xy must be identical, which gives us conditions for determining m and n. Substituting for z in (1) and (3) its value from (2), we get

$$(P + P''m^2)x^2 + (P' + P''n^2)y^2 + 2P''mnxy = H \text{......} (4),$$

$$(1 + m^2)x^2 + (1 + n^2)y^2 + 2mnxy = r^2 \text{......} (5).$$

Comparing each term separately, those involving xy will coincide if either

$$m = 0, \quad n = 0, \quad \text{or} \quad r^2 = \frac{H}{P''}.$$

Taking the first condition and comparing the other terms,

$$\frac{H}{P} = r^2, \text{ and } \frac{P' + P''n^2}{H} = \frac{1 + n^2}{r^2},$$

which gives $P' + P''n^2 = P(1 + n^2)$,

$$\text{and } n = \pm\sqrt{\frac{P - P'}{P'' - P}}.$$

If we suppose $n = 0$, we find in the same manner

$$m = \pm\sqrt{\frac{P' - P}{P'' - P'}}.$$

The third condition leads to no result, and therefore is not to be considered.

In the ellipsoid, P, P′, P″ are all positive, and

$$P < P' < P''.$$

This shows that the value of n is impossible, and that of m possible; therefore there are two directions arising from the doubtful sign in which the ellipsoid may be cut in circular sections, determined by the equation to the cutting plane,

$$z = \pm\sqrt{\frac{P' - P}{P'' - P'}}\,x.$$

In the hyperboloid of one sheet P″ is negative, and the value of n is possible and m impossible. In the hyperboloid of two sheets P′ and P″ are both negative, and $P'' < P'$, m is possible and n impossible. It is true, that for a plane passing through the centre the section is impossible, but a plane drawn parallel to this at a sufficient distance from the centre will cut the surface in a circle.

The equation to the surfaces without a centre is

$$p'y^2 + pz^2 = pp'x.$$

Let this be cut by a plane

$$x = mz + ny,$$

which also cuts the sphere

$$x^2 + y^2 + z^2 = 2rx.$$

The equations to the projections of the sections on the plane of zy are

$$p'y^2 + pz^2 - mpp'z - npp'y = 0,$$

$$(1 + n^2)\,y^2 + (1 + m^2)\,z^2 + 2mnzy - 2mrz - 2nry = 0.$$

In order that these may coincide, the term involving zy must vanish, which will be the case if $m = 0$ or $n = 0$.

$$\text{If } m = 0, \text{ then } 1 + n^2 = \frac{p'}{p} \text{ and } n = \pm\sqrt{\frac{p' - p}{p}}.$$

$$\text{If } n = 0, \text{ then } 1 + m^2 = \frac{p}{p'} \text{ and } m = \pm\sqrt{\frac{p - p'}{p'}}.$$

In the elliptic paraboloid p and p' are both positive, and according as p' is greater or less than p, the first or second is to be taken, the other becoming impossible. In either case there are two series of circular sections corresponding to the positive and negative sign.

In the hyperbolic paraboloid p or p' is negative, so that there are no sections in which it is cut in a circle. This would appear also from the nature of the surface, as it can never be cut by a plane in a closed curve.

The same method may be applied to the oblique cone, so as to determine the sub-contrary sections.

By determining the circular sections in this manner, it is seen at once that any two belonging to different series are situate on the same sphere.

F.

III.—PROPOSITIONS IN THE THEORY OF NUMBERS.

The proposition, "The continued product of any m consecutive integers is divisible by $1.2.3...m$," has been strictly proved by induction; but this mode of proof only shows, if we may so speak, that the proposition *must* be true, without showing *why* it is true.

We have seen it reasoned, that since the product of any m consecutive integers is divisible by each of the natural numbers up to m separately, it is divisible by their product: but a little consideration will show that this does not follow.

The problem from which the above proposition is here deduced, has been proposed and solved before, but not, that we are aware, in the same form.

Prop. 1. If p be a prime number, the index of the highest power of p that is a divisor of the continued product $1.2.3...M$, is $\dfrac{M - S(M)}{p-1}$, where $S(M)$ represents the sum of the digits of M expressed in the scale, whose radix is p.

We shall adopt the notation used by several Continental mathematicians, $M!$ for $1.2.3...M$.

Let the quotient of M by p be M_1, and the remainder a_0, then

$$M = M_1 p + a_0,$$

and the multiples of p in the series 1, 2, 3, &c. M, are

$$p,\ 2p,\ 3p,\ \text{\&c. } M_1 p;$$

wherefore, if K denote the product of the remaining numbers in the former series,

$$M! = K \times p.2p.3p...M_1 p,$$
$$= Kp^{M_1}.M_1!$$

Let r be the index of the greatest power of p in $M!$,
r_1 .... $M_1!$,
then from the last equation

$$r = M_1 + r_1.$$

For the same reason, if

$$M_1 = M_2 p + a_1, \text{ and } M_2 = M_3 p + a_2, \text{ \&c.}$$

and r_2, r_3, &c. be corresponding quantities for $M_2!$, $M_3!$, &c.

$$\begin{aligned} r_1 &= M_2 + r_2, \\ r_2 &= M_3 + r_3, \\ &\cdots\cdots\cdots\cdots \\ r_{m-1} &= M_m + r_m, \\ r_m &= 0, \end{aligned}$$

M_m being that quotient less than p at which we must finally arrive. Adding all these equations,

$$r = M_1 + M_2 + M_3 + \ldots\ldots + M_m.$$

But, adding the equations

$$\begin{aligned} M &= M_1 p + a_0, \\ M_1 &= M_2 p + a_1, \\ M_2 &= M_3 p + a_2, \\ &\cdots\cdots\cdots\cdots \\ M_{m-1} &= M_m p + a_{m-1}, \\ M_m &= a_m; \end{aligned}$$

and observing that a_0, a_1, &c. a_m are the successive digits of M expressed in the scale of p, we have

$$M + M_1 + M_2 + \ldots + M_m = (M_1 + M_2 + \ldots + M_m)\, p + S(M);$$

$$\text{whence } M_1 + M_2 + \ldots + M_m = \frac{M - S(M)}{p - 1},$$

and the proposition is proved.

The following proposition is also useful.

PROP. 2. If M and N be two numbers, of which M is the greater, and S(M), S(N) the sums of their digits to any radix r, S(M – N) is not less than S(M) – S(N).

For let $M = a_m r^m + a_{m-1} r^{m-1} + \ldots + a_n r^n + a_{n-1} r^{n-1} + \ldots,$
and $N = \qquad b_n r^n + b_{n-1} r^{n-1} + \ldots,$
then

$$M - N = a_m r^m + a_{m-1} r^{m-1} + \ldots + (a_n - b_n) r^n + (a_{n-1} - b_{n-1}) r^{n-1} + \ldots$$

Now, as long as the digits of N are not greater than the corresponding digits of M, the digits of M – N will be the excesses of those of M over those of N. But suppose b_x greater than a_x, then we form the digits of M – N by writing that part of the difference thus,

$$(a_{x+1} - b_{x+1} - 1)\, r^{x+1} + (r + a_x - b_x)\, r^x,$$

and the sum of these two digits is

$$a_{s+1} - b_{s+1} + a_s - b_s + r - 1;$$

hence $S(M - N)$ may be greater, but cannot be less, than

$$S(M) - S(N).$$

PROP. 3. The product of any m consecutive integers is divisible by $1.2.3 .. m$.

By Prop. 1, the index of p in $N!$ is $\dfrac{N - S(N)}{p-1}$, and in $(N+m)!$, $\dfrac{N + m - S(N+m)}{p-1}$; hence the index of p in

$$(N+1)(N+2)\ldots(N+m) \text{ or } \frac{(N+m)!}{N!},$$

is the difference of these quantities, or

$$\frac{m - S(N+m) + S(N)}{p-1}.$$

The index of p in $m!$ is $\dfrac{m - S(m)}{p}$; and, by Prop. 2, making

$$M = N + m, \quad S(m) \text{ is not } < S(N+m) - S(N),$$

$$\text{therefore } m - S(m) \text{ is not } > m - S(N+m) + S(N);$$

and the index of p in $m!$ is not greater than the index of p in $(N+1)(N+2)\ldots(N+m)$. Hence, if

$$(N+1)(N+2)\ldots(N+m) = 2^a.3^b.5^c\ldots,$$

$$\text{and } 1.2.3\ldots m = 2^\alpha.3^\beta.5^\gamma\ldots,$$

α is not $> a$, β not $> b$, γ not $> c$, &c. Consequently

$$(N+1)(N+2)\ldots(N+m) \text{ is divisible by } 1.2.3\ldots m.$$

S. S. G.

IV.—NOTES ON FOURIER'S HEAT.

THE method employed by Fourier to integrate the partial differential equations which occur in the Theory of Heat, is to assume some simple form of a singular solution, and afterwards to extend it so as to include all the circumstances of the problem. It is in effecting this that he has displayed the great resources of his analysis, and imparted so great an interest to his work by the variety and ingenuity of his methods. Indeed there is a freshness and originality in the writings of Fourier which make them in no ordinary degree arrest the attention of the reader. But however much we may admire the means by which Fourier has overcome the difficulties of the problems he had to deal with, yet it seems more

agreeable to the usual style of mathematical investigation to deduce a result by limiting the general solution by means of the conditions of the problem, than by extending a particular case.

That this may be sometimes done with even more readiness than by Fourier's method, will be seen by the following solution of a problem given in p. 161 of the *Théorie de la Chaleur.* We may remark, that there is in general no difficulty in the solution of the partial differential equations, but only in the proper determination of the arbitrary functions in the solution, so as to suit the conditions of the problem.

If a rectangular plate, bounded by two infinite parallel edges, have one of its extremities kept at a constant temperature 1, while the infinite edges perpendicular to the heated edge are retained at a constant temperature 0, the equation from which the temperature is to be determined is

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where v is the temperature at the point x, y, the origin being at the middle point of the heated edge, the axis of x bisecting the plate, and the axis of y parallel to the heated edge. For the sake of shortness Fourier represents the breadth of the plate by π.

The solution of the equation (1) by the method of the separation of the symbols of operation from those of quantity, is

$$v = \cos\left(y\frac{d}{dx}\right)\phi(x) + \sin\left(y\frac{d}{dx}\right)\psi(x) \;\ldots\ldots (2),$$

$\phi(x)$ and $\psi(x)$ being arbitrary functions of x. And it may also be put under the form

$$v = \mathrm{F}(x + y\sqrt{-1}) + f(x - y\sqrt{-1}),$$

where $\mathrm{F}(x) = \frac{1}{2}\{\phi(x) + \psi(x)\}$ and $f(x) = \frac{1}{2}\{\phi(x) - \psi(x)\}$.

Now, on looking at the circumstances of the problem, it will be seen that it must be subject to the following conditions:

1st. v must be symmetrical with regard to y and $-y$.

2nd. $v = 0$ when $y = \frac{\pi}{2}$ or $-\frac{\pi}{2}$, whatever x may be.

3rd. $v = 1$ when $x = 0$, whatever y may be.

4th. v must be very small when x is very large.

From the first condition we must have $\psi(x) = 0$, as otherwise the second term would change its sign when $-y$ is put for y. Hence we have only

$$v = \cos\left(y\frac{d}{dx}\right)\phi(x) \;\ldots\ldots\ldots (3).$$

By the second condition, putting $\frac{\pi}{2}$ for y in equation (3), we have

$$0 = \cos\left(\frac{\pi}{2}\frac{d}{dx}\right)\phi(x) \;\ldots\ldots\ldots (4).$$

Now this is in fact a linear differential equation with constant coefficients, and of an infinite order. By the principles laid down in Art. V. of our first Number, we can integrate this equation if we know the roots of the equation $\cos\left(\frac{\pi}{2}z\right) = 0$. Now these are

$$\pm 1, \ \pm 3, \ \pm 5, \ \&c.$$

being in number infinite. Hence the solution of (4) is

$$\phi(x) = \left\{\begin{array}{l} C_1\epsilon^{-x} + C_3\epsilon^{-3x} + C_5\epsilon^{-5x} + \&c. \\ + C'_1\epsilon^{x} + C'_3\epsilon^{3x} + C'_5\epsilon^{5x} + \&c. \end{array}\right\} \ldots\ldots(5),$$

the number of terms and arbitrary constants being infinite. By the fourth condition it appears that the second line of (5) must disappear, as otherwise v would be very large when x is very large. Hence we must have

$$C'_1 = 0, \ C'_3 = 0, \ C'_5 = 0, \ \&c.$$

and equation (5) is reduced to

$$\phi(x) = C_1\epsilon^{-x} + C_3\epsilon^{-3x} + C_5\epsilon^{-5x} + \&c\ldots\ldots(6),$$

and v becomes

$$v = \cos\left(y\frac{d}{dx}\right)(C_1\epsilon^{-x} + C_3\epsilon^{-3x} + C_5\epsilon^{-5x} + \&c.)\ldots\ldots(7).$$

By the third condition $v = 1$ when $x = 0$. If then we expand the symbol of operation in (7), operate on each term separately, and then make $x = 0$, we shall find

$$1 = C_1\cos y + C_3\cos 3y + C_5\cos 5y + \&c\ldots\ldots(8),$$

where y is contained between the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

In order to determine the arbitrary constants, we shall follow Fourier's method of definite integrals. If we multiply both sides of (8) by $\cos y\, dy$, and integrate between the limits $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$, all the terms except the first will disappear, as they can each be decomposed into the cosines of even multiples of y, which, on integration, vanish at both limits. Hence we have

$$\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} dy\cos y = C_1\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} dy\cos^2 y = \frac{C_1}{2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} dy\,(1 + \cos 2y),$$

whence we find $C_1 = \frac{4}{\pi}$.

In a similar manner we should find

$$C_3 = -\frac{1}{3}\frac{4}{\pi}, \quad C_5 = -\frac{1}{5}\frac{4}{\pi}, \quad C_7 = -\frac{1}{7}\frac{4}{\pi},$$

and so on.

Substituting these values in equation (7), it becomes

$$\frac{\pi}{4} v = \cos\left(y \frac{d}{dx}\right)\left(\varepsilon^{-x} - \frac{1}{3}\varepsilon^{-3x} + \frac{1}{5}\varepsilon^{-5x} - \frac{1}{7}\varepsilon^{-7x} + \&c.\right).$$

Now if we expand the sign of operation, and apply it to such a term as ε^{-nx}, we shall find that it becomes $\cos ny\, \varepsilon^{-nx}$. Hence the expression for v becomes

$$\frac{\pi}{4} v = \varepsilon^{-x}\cos y - \tfrac{1}{3}\varepsilon^{-3x}\cos 3y + \tfrac{1}{5}\varepsilon^{-5x}\cos 5y - \&c.$$

which is one form of the solution which Fourier gives. It may easily be reduced to a more simple form. For if we substitute for the cosines their exponential values, we have

$$\frac{\pi}{2} v = \begin{cases} \varepsilon^{-(x-y\sqrt{-1})} - \frac{1}{3}\varepsilon^{-3(x-y\sqrt{-1})} + \frac{1}{5}\varepsilon^{-5(x-y\sqrt{-1})} - \&c. \\ + \varepsilon^{-(x+y\sqrt{-1})} - \frac{1}{3}\varepsilon^{-3(x+y\sqrt{-1})} + \frac{1}{5}\varepsilon^{-5(x+y\sqrt{-1})} - \&c. \end{cases}$$

which, by Gregorie's series, become

$$\frac{\pi}{2} v = \tan^{-1}\varepsilon^{-(x-y\sqrt{-1})} + \tan^{-1}\varepsilon^{-(x+y\sqrt{-1})}$$

$$= \tan^{-1}\frac{\varepsilon^{-(x-y\sqrt{-1})} + \varepsilon^{-(x+y\sqrt{-1})}}{1 - \varepsilon^{-2x}}$$

$$= \tan^{-1}\left(\frac{2\cos y}{\varepsilon^{x} - \varepsilon^{-x}}\right),$$

which is the simplest form that the expression can assume.

D. F. G.

V.—DEMONSTRATIONS OF SOME PROPERTIES OF SURFACES OF THE SECOND ORDER.

To the Editor of the Cambridge Mathematical Journal.

Sir,—I observe the author of a Paper in your first Number says, he is not aware that any person has made use of the symmetrical form of the equations to a straight line: he will find them employed in Cauchy's "*Leçons sur les Applications du Calcul Infinitésimal.*"

Perhaps the following solutions of two common problems may be acceptable to some of your readers.

1. To find the locus of the bisections of parallel chords in the surface whose equation is

$$Ax^2 + By^2 + Cz^2 = k.$$

Suppose any one of the chords meets the surface in the points whose coordinates are $x_0, y_0, z_0, x_1, y_1, z_1$; then if α, β, γ are the angles which the chord makes with the three axes, we have

$$\frac{x_0 - x_1}{\cos\alpha} = \frac{y_0 - y_1}{\cos\beta} = \frac{z_0 - z_1}{\cos\gamma} \text{ ......(1).}$$

Let x, y, z be the coordinates of the middle point of the chord; then

$$x = \frac{x_0 + x_1}{2}, \quad y = \frac{y_0 + y_1}{2}, \quad z = \frac{z_0 + z_1}{2},$$

$$\text{or } \frac{x_0 + x_1}{x} = \frac{y_0 + y_1}{y} = \frac{z_0 + z_1}{z};$$

therefore, combining these equations with (1), we have

$$\frac{x_0^2 - x_1^2}{x\cos\alpha} = \frac{y_0^2 - y_1^2}{y\cos\beta} = \frac{z_0^2 - z_1^2}{z\cos\gamma} \text{ ......(2).}$$

Now, from the equation to the surface we have

$$A(x_0^2 - x_1^2) + B(y_0^2 - y_1^2) + C(z_0^2 - z_1^2) = 0,$$

which, combined with (2), gives immediately

$$Ax\cos\alpha + By\cos\beta + Cz\cos\gamma = 0,$$

which is the equation to the locus required.

2. To find the locus of the intersection of tangent planes drawn parallel to any system of conjugate diameters in the same surface.

I shall first observe, that the projection of the diagonal of a parallelogram on any line whatever, is evidently equal to the sum of the projections of any two contiguous sides on the same line. Whence it easily follows, that the projection of the diagonal of a parallelopiped is equal to the sum of the projections of any three of its edges, of which no two are parallel to each other.

Hence, if ξ, η, ζ are three straight lines drawn from the origin to three points P_0, P_1, P_2, whose coordinates are $x_0, y_0, z_0, x_1, y_1, z_1, x_2, y_2, z_2$, and if through the points P_0, P_1, P_2 we draw planes parallel respectively to the planes of $\eta\zeta, \xi\zeta, \xi\eta$, these three planes will intersect in some point P, and a line drawn from the origin to P will be the diagonal of a parallelopiped, of which ξ, η, ζ are three edges. And if we call x, y, z the coordinates of P, and project the diagonal and the three lines ξ, η, ζ on the three axes successively, we have evidently, by the principle above mentioned,

$$\left.\begin{aligned} x &= x_0 + x_1 + x_2 \\ y &= y_0 + y_1 + y_2 \\ z &= z_0 + z_1 + z_2 \end{aligned}\right\} \text{......(3).}$$

Now suppose ξ, η, ζ are three semiconjugate diameters of the surface proposed, we have of course

$$Ax_1x_2 + By_1y_2 + Cz_1z_2 = 0,$$
$$Ax_0x_2 + By_0y_2 + Cz_0z_2 = 0,$$
$$Ax_0x_1 + By_0y_1 + Cz_0z_1 = 0;$$

and if we multiply these equations by 2, and add them to the three following, viz.

$$Ax_0^2 + By_0^2 + Cz_0^2 = k,$$
$$Ax_1^2 + By_1^2 + Cz_1^2 = k,$$
$$Ax_2^2 + By_2^2 + Cz_2^2 = k,$$

we obtain immediately

$$A(x_0+x_1+x_2)^2 + B(y_0+y_1+y_2)^2 + C(z_0+z_1+z_2)^2 = 3k,$$

or by (3)

$$Ax^2 + By^2 + Cz^2 = 3k,$$

which is evidently the equation to the locus required.

I do not know whether this solution is new, but I do not recollect to have seen it before.

The equations (3) lead immediately to the common formulæ for the transformation of coordinates; for if we call $\alpha_0, \beta_0\ \gamma_0, \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$, the angles which ξ, η, ζ make with the three axes respectively, we have $x_0 = \xi \cos \alpha_0$, $y_0 = \xi \cos \beta_0$, $z_0 = \xi \cos \gamma_0$, &c. and the equations (3) become

$$x = \xi \cos \alpha_0 + \eta \cos \alpha_1 + \zeta \cos \alpha_2,$$
$$y = \xi \cos \beta_0 + \eta \cos \beta_1 + \zeta \cos \beta_2,$$
$$z = \xi \cos \gamma_0 + \eta \cos \gamma_1 + \zeta \cos \gamma_2,$$

where ξ, η, ζ are evidently the coordinates of P, referred to lines drawn from the origin through P_0, P_1, P_2 as axes.

I am, Sir, &c. M. N. N.

Oxford, Jan. 31, 1838.

VI.—ON GENERAL DIFFERENTIATION.—No. II.

In our first Number we investigated the general differential coefficients of ε^{nx} and x^n. We shall now proceed to find those of other simple functions, and to give some examples of the application of the theory.

1. We found that our formulæ gave an infinite value for $\frac{d^\alpha x^n}{dx^\alpha}$, when n is not, and $n-\alpha$ is, a positive integer. But by using the complementary function, a finite value of another form may be obtained, in the same manner as the value of $\int \frac{dx}{x}$ may be derived from that of $\int x^n dx$.

Let $\alpha - n$ in formula (N), and $n - \alpha$ in formula (O), (p. 19), be assumed equal to $r + \beta$, when r is an integer, then the second

member of each will be a multiple of $\frac{x^{r+\beta}}{\sin(r+\beta)\pi}$, which is infinite when $\beta=0$, but by taking in a part of the complementary function becomes $\frac{x^r(x^\beta - a^\beta)}{\sin(r+\beta)\pi}$, which is a vanishing fraction whose value is $\frac{(-1)^r}{\pi} x^r \log\frac{x}{a}$ when $\beta=0$. Hence in these cases (N) and (O) become respectively

$$\frac{d^{-a}}{dx^{-a}}\frac{1}{x^n} = (-1)^{1+n}\frac{1}{\Gamma(1+a-n)\Gamma(n)} x^{a-n}\log\frac{x}{a} \;\ldots\; \text{(Q)},$$

$$\frac{d^a x^n}{dx^a} = (-1)^n \frac{\sin n\pi}{\pi}\cdot\frac{\Gamma(1+n)}{\Gamma(1+n-a)} x^{n-a}\log\frac{x}{a} \;\ldots\; \text{(R)}.$$

2. The value of $\frac{d^a \log x}{dx^a}$ may be readily derived from the formulæ for the differential coefficients of powers of x. For since

$$\frac{d}{dx}\log x = \frac{1}{x},$$

$$\frac{d^a}{dx^a}\log x = \frac{d^{a-1}}{dx^{a-1}}\frac{1}{x}.$$

Hence, if a be positive,

$$\frac{d^a}{dx^a}\log x = (-1)^{a-1}\Gamma(a)\frac{1}{x^a}, \text{ by (E)} \;\ldots\ldots\ldots\; \text{(S)},$$

and
$$\frac{d^{-a}}{dx^{-a}}\log x = (-1)^a\frac{\pi}{\sin a\pi . \Gamma(1+a)} x^a, \text{ by (N)} \;\ldots\ldots\; \text{(T)}.$$

3. If it be known that

$$\frac{d^a}{dx^a}\phi(x) = \psi(x),$$

it is evident that

$$\frac{d^a}{dx^a}\phi(mx+a) = m^a\psi(mx+a);$$

hence the general differential coefficients of all rational functions of x can be found, by preparing them as for integration. Also, if the differential coefficient of a function to any integral index be a rational function, its general differential coefficient may be found. As an example, let

$$y = \tan^{-1} x,$$

then
$$\frac{dy}{dx} = \frac{1}{1+x^2} = \tfrac{1}{2}\left(\frac{1}{1+\sqrt{-1}\,x} + \frac{1}{1-\sqrt{-1}\,x}\right);$$

therefore, using the formula (P),

$$\frac{d^a y}{dx^a} = \tfrac{1}{2}\frac{P(-1)}{P(-a)}\left\{\frac{(\sqrt{-1})^a}{(1+\sqrt{-1}\,x)^a} + \frac{(-\sqrt{-1})^a}{(1-\sqrt{-1}\,x)^a}\right\}.$$

Substituting for x its value $\tan y$, observing that

$$\sqrt{-1} = \cos(2r + \tfrac{1}{2})\pi + \sqrt{-1}\sin(2r + \tfrac{1}{2})\pi,$$

and reducing by De Moivre's theorem, we obtain

$$\frac{d^a y}{dx^a} = \frac{P(-1)}{P(-a)}(\cos y)^a \cos a\,\{y - (2r + \tfrac{1}{2})\pi\}.$$

The value of $\dfrac{P(-1)}{P(-a)}$ is that of $x^a.\dfrac{d^{a-1}x^{-1}}{dx^{a-1}}$, which is

$$(-1)^{a-1}\,\Gamma(a)$$

when a is positive, and

$$(-1)^a \frac{\pi}{\sin(-a\pi).\Gamma(1-a)}$$

when a is negative. The formula fails when a is a negative integer.

4. Let us ascertain in what cases the differential coefficient of a constant quantity is not zero.

Since $C = Cx^0$,

$$\frac{d^a C}{dx^a} = C\,\frac{P(0)}{P(-a)}\,x^{-a} = \frac{C}{P(-a)}\,x^{-a},$$

which will not be zero if $P(-a)$ be not infinite, which is only when $-a$ is a positive integer, or a a negative integer, that is, in the case of common integration.

5. The differential coefficient of infinity may be finite. For $\infty = CP(m)\,x^n$, if m be not a positive integer: therefore

$$\frac{d^a \infty}{dx^a} = C\,\frac{P(m)\,P(n)}{P(n-a)}\,x^{n-a},$$

which will be finite if n be a positive integer, and $n - a$ not.

6. We proceed to find the general differential coefficients of $\cos x$ and $\sin x$.

Since $\cos x + \sqrt{-1}\sin x = \varepsilon^{\sqrt{-1}\,x}$

$$\frac{d^a \cos x}{dx^a} + \sqrt{-1}\,\frac{d^a \sin x}{dx^a} = (\sqrt{-1})^a\,\varepsilon^{\sqrt{-1}\,x}$$

$$= \{\cos(2r + \tfrac{1}{2})\pi + \sqrt{-1}\sin(2r + \tfrac{1}{2})\pi\}^a(\cos x + \sqrt{-1}\sin x)$$

$$= \cos\{(2r + \tfrac{1}{2})a\pi + x\} + \sqrt{-1}\sin\{(2r + \tfrac{1}{2})a\pi + x\}.$$

In like manner,

$$\frac{d^a \cos x}{dx^a} - \sqrt{-1}\,\frac{d^a \sin x}{dx^a}$$

$$= \cos\{(2r' + \tfrac{1}{2})a\pi + x\} - \sqrt{-1}\sin\{(2r' + \tfrac{1}{2})a\pi + x\},$$

r' being some integer, not necessarily the same as r. Adding and subtracting,

$$\left.\begin{aligned}
&\frac{d^a \cos x}{dx^a} \\
&= \{\cos (r - r')\, a\pi + \sqrt{-1} \sin (r - r')\, a\pi\} \\
&\qquad\qquad \cos \{(r + r' + \tfrac{1}{2})\, a\pi + x\}, \\
&\frac{d^a \sin x}{dx^a} \\
&= \{\cos (r - r')\, a\pi + \sqrt{-1} \sin (r - r')\, a\pi\} \\
&\qquad\qquad \sin \{(r + r') + \tfrac{1}{2})\, a\pi + x\},
\end{aligned}\right\} \ldots\ldots (\mathrm{U}).$$

Since r and r' are both arbitrary, there is no relation between $r + r'$ and $r - r'$, except that they must be both odd or both even. If a be a rational fraction whose denominator, when in its lowest terms, is n, $r + r'$ must go through all the $2n$ values,

$$0, 1, 2, 3, \ldots\ldots 2n - 1,$$

before the variety of values of the trigonometrical expression into which it enters can be exhausted; and the same is true of $r - r'$. Now any of the n odd values of $r + r'$ may be combined with any of the n odd values of $r - r'$, and thus n^2 different values of the total expression are found. The even values of $r + r'$ and $r - r'$ produce as many more different values, hence the fractional differential coefficient of each of the quantities $\cos x$ and $\sin x$ has $2n^2$ different values.*

For example, the differential coefficients to the index $\frac{1}{2}$ of $\cos x$ and $\sin x$ have 2×2^2 or 8, values. Those of the former will be found to be

$$\pm \cos\left(\frac{\pi}{4} + x\right), \quad \pm \sqrt{-1} \cos\left(\frac{3\pi}{4} + x\right),$$

$$\pm \cos\left(\frac{5\pi}{4} + x\right), \quad \pm \sqrt{-1} \cos\left(\frac{7\pi}{4} + x\right).$$

The reader may observe that if differentiation to the index $\frac{1}{2}$ be performed twice in succession upon $\cos x$ by any one of these formulæ, provided the same be used both times, the result will always be $-\sin x$, as it should be.

7. The same multiplicity of values attends the differential coefficients of other trigonometrical functions. For instance, let us consider the functions $\varepsilon^{mx} \cos nx$ and $\varepsilon^{mx} \sin nx$. It will be convenient to asume $m = a \cos \theta$, and $n = a \sin \theta$. We may then change ax into x, and they become

$$\varepsilon^{x \cos \theta} \cos (x \sin \theta), \quad \varepsilon^{x \cos \theta} \sin (x \sin \theta),$$

which we shall call Θ and Θ' respectively, as Mr. R. Murphy has

* M. Liouville did not employ the direct process which we have, and consequently did not discover that the differential coefficients had more than $2n$ values.

done in the *Camb. Phil. Trans.* vol. v, where he has proved some properties of them. We have then

$$\Theta + \sqrt{-1}\,\Theta' = \varepsilon^{x\cos\theta}\{\cos(x\sin\theta) + \sqrt{-1}\sin(x\sin\theta)\}$$
$$= \varepsilon^{x(\cos\theta + \sqrt{-1}\sin\theta)};\ \text{therefore}$$

$$\frac{d^a}{dx^a}(\Theta + \sqrt{-1}\,\Theta') = (\cos\theta + \sqrt{-1}\sin\theta)^a\,\varepsilon^{x(\cos\theta + \sqrt{-1}\sin\theta)}$$

$$= \{\cos a(2r\pi + \theta) + \sqrt{-1}\sin a(2r\pi + \theta)\}$$
$$\varepsilon^{x\cos\theta}\{\cos(x\sin\theta) + \sqrt{-1}\sin(x\sin\theta)\}$$
$$= \varepsilon^{x\cos\theta}\{\cos(x\sin\theta + 2ar\pi + a\theta) + \sin(x\sin\theta + 2ar\pi + a\theta)\}.$$

In like manner

$$\frac{d^a}{dx^a}(\Theta - \sqrt{-1}\,\Theta') =$$
$$\varepsilon^{x\cos\theta}\{\cos(x\sin\theta + 2ar'\pi + a\theta) - \sin(x\sin\theta + 2ar'\pi + a\theta)\}.$$

By addition and subtraction

$$\left.\begin{aligned}
&\frac{d^a\Theta}{dx^a} = \\
&\{\cos(r - r')\,a\pi + \sqrt{-1}\sin(r - r')\,a\pi\} \\
&\qquad \varepsilon^{x\cos\theta}\cos\{x\sin\theta + (r + r')\,a\pi + a\theta\}, \\
&\frac{d^a\Theta'}{dx^a} = \\
&\{\cos(r - r')\,a\pi + \sqrt{-1}\sin(r - r')\,a\pi\} \\
&\qquad \varepsilon^{x\cos\theta}\sin\{x\sin\theta + (r + r')\,a\pi + a\theta\},
\end{aligned}\right\} \ldots\ldots (V).$$

As before, the number of values of each of these expressions is twice the square of the denominator of a, supposed a rational fraction in its lowest terms.

The necessity of supposing r and r' different, may be proved as follows. If θ be changed into $-\theta$, Θ remains the same and Θ' is altered only in sign: no new value of the differential coefficient ought therefore to be found by making this change: but that would be the case if r and r' had been supposed equal: therefore that supposition would not give all the values of the differential coefficients. A similar remark applies to the last article.

8. We now come to another division of our subject, in which we shall prove certain formulæ discovered by M. Liouville, and applied by him to the solution of a variety of problems. The first of these is

$$\int_0^\infty \phi(x + a)\,a^{n-1}\,da = (-1)^n\,\Gamma(n)\,\frac{d^{-n}}{dx^{-n}}\,\phi(x) \ldots\ldots\ldots (W).$$

This may readily be proved by the theory of generating functions.

Let $G\phi(x)$ denote the generating function of $\phi(x)$, or that function of t in the expansion of which the coefficient of t^x is $\phi(x)$; then we have

$$G\phi(x+a) = t^{-a}G\phi(x),$$

$$G\int_0^\infty \phi(x+a)\,a^{n-1}\,da = \int_0^\infty t^{-a}\,a^{n-1}\,da.G\,\phi(x).$$

Let $t^a = \varepsilon^\theta$, then $a = \dfrac{\theta}{\log t}$, and the second member of the preceding equation becomes

$$\Gamma(n)\,(\log t)^{-n}\,G\,\phi(x);$$

$$\text{but } \left(\log \frac{1}{t}\right)^r G\phi(x) = G\,\frac{d^r}{dx^r}\,\phi(x):$$

$$\text{therefore } G\int_0^\infty \phi(x+a)\,a^{n-1}\,da = (-1)^n\,\Gamma(n)\,G\,\frac{d^{-n}}{dx^{-n}}\,\phi(x),$$

$$\text{and } \int_0^\infty \phi(x+a)\,a^{n-1}da = (-1)^n\,\Gamma(n)\,\frac{d^{-n}}{dx^{-n}}\,\phi(x).$$

It may be remarked, that if the form of $\phi(x)$ be such that the first side is infinite, the second may be made to agree with it by the aid of the complimentary function, if it do not without; so that the formula is true for all forms of $\phi(x)$.

9. An obvious use of this formula is to find the form of $\phi(x)$ when the value of the definite integral is known, as in the following problem.

Each particle M of a line AB, infinite in both directions, exerts upon a particle P without it a force perpendicular to the plane ABP, and proportional to sin PMA, and to an unknown function $F(r)$ of the distance PM or r. The total action of the line upon P is a known function $f(y)$ of the perpendicular distance y of P from AB. It is required to find the function $F(r)$.

This is an important physical problem, for if AB represent a voltaic current, and P a pole of a magnet, the action of M on P is such as has been described; and the total action is found by experiment to be inversely as y.

Let C be the foot of the perpendicular from P on AB, and let $CM = z$, then $r^2 = y^2 + z^2$, and $\sin PMA = \dfrac{y}{r}$, whence the whole action of AB is

$$\int_{-\infty}^{+\infty} \frac{y\,F\sqrt{(y^2+z^2)}}{\sqrt{y^2+z^2}}\,dz = 2y\int_0^\infty \frac{F\sqrt{(y^2+z^2)}}{\sqrt{y^2+z^2}}\,dz,$$

which is equal to $f(y)$ by hypothesis.

Assume $\dfrac{\mathrm{F}(r)}{r} = \phi(r^2)$, therefore

$$2\int_0^\infty \phi(y^2 + z^2)\, dz = \frac{f(y)}{y}.$$

Let $y^2 = x$, $z^2 = a$, then the first member becomes

$$\int_0^\infty \phi(x + a)\, a^{-\frac{1}{2}}\, da,$$

which by (W) is equal to

$$\sqrt{-1}\ \Gamma(\tfrac{1}{2}) \left(\frac{d}{dx}\right)^{-\frac{1}{2}} \phi(x);$$

$$\text{therefore } \sqrt{-1}\ \Gamma(\tfrac{1}{2}) \left(\frac{d}{dx}\right)^{-\frac{1}{2}} \phi(x) = \frac{f(\sqrt{x})}{\sqrt{x}},$$

$$\text{and } \phi(x) = \frac{1}{\sqrt{-1}\ \Gamma(\frac{1}{2})} \left(\frac{d}{dx}\right)^{\frac{1}{2}} \frac{f(\sqrt{x})}{\sqrt{x}}.$$

Supposing that the differentiation is performed, and r^2 is substituted for x, $\mathrm{F}(r)$, which is equal to $r\,\phi(r^2)$, is known.

If $f(y) = \dfrac{a}{y}$, $\dfrac{f(\sqrt{x})}{\sqrt{x}} = \dfrac{a}{x}$, and by formula (E), p. 14, we find

$$\phi(x) = \tfrac{1}{2} a x^{-\frac{3}{2}};$$

$$\text{therefore } \mathrm{F}(r) = \frac{a}{2r^2}.$$

It is easy to verify this result by ordinary integration, and thus to confirm the truth of the principles of the new calculus.

10. It will be convenient to possess a formula in which the limits of the integral shall be 0 and 1. Such a one may de obtained from (W) by changing the independent variable.

Let $x + a = \dfrac{x}{\theta}$; then when $a=0$, $\theta=1$, and when $a=\infty$, $\theta=0$: also, $a = \dfrac{1-\theta}{\theta} \cdot x$, $da = -\dfrac{x d\theta}{\theta^2}$, and the formula is changed into

$$\int_0^1 \phi\left(\frac{x}{\theta}\right) \frac{x^n}{\theta^{n+1}} (1-\theta)^{n-1}\, d\theta = (-1)^n \Gamma(n) \frac{d^{-n}}{dx^{-n}} \phi(x).$$

Let $x^{n+1} \phi(x) = \psi\left(\dfrac{1}{x}\right)$; therefore

$$\int_0^1 \psi\left(\frac{\theta}{x}\right) (1-\theta)^{n-1}\, d\theta = (-1)^n \Gamma(n)\, x \frac{d^{-n}}{dx^{-n}} \left\{ \frac{1}{x^{n+1}} \psi\left(\frac{1}{x}\right) \right\} \ldots (\mathrm{X}).$$

The following modification of the formula is often convenient.

Assume $\theta = \beta x$ and $x = \dfrac{1}{y}$, then we have

$$\int_0^y \psi(\beta)(y-\beta)^{n-1} d\beta = (-1)^n \Gamma(n) y^{n-1} d^{-n} \{y^{n+1} \psi(y)\} \left(d.\frac{1}{y}\right)^n \ldots (\mathrm{Y}).$$

The differentiation indicated in the second member must be performed with respect to $\frac{1}{y}$ as the independent variable.

11. As an example of the use of the last formula, we select the following problem, which includes, as a particular case, that of finding the tantochronous curve.

To find the curve such that the time occupied by the descent of a particle sliding along it by the action of gravity to a given point from a given vertical height above it, shall be a given function of that height.

Let x be measured vertically. Then the known expression for the time of descent from a height h to the origin, is

$$\int_0^h \frac{1}{\sqrt{2g(h-x)}}\,\frac{ds}{dx}dx;$$

but by hypothesis this is equal to the known function $f(h)$. Putting $\frac{ds}{dx} = \psi(x)$, and comparing the integral with that in formula (Y), we find

$$\sqrt{-1}\,\Gamma(\tfrac{1}{2})\,h^{-\frac{1}{2}}\,d^{-\frac{1}{2}}\{h^{\frac{3}{2}}\,\psi(h)\}\left(d.\frac{1}{h}\right)^{\frac{1}{2}} = \sqrt{2g}\,f(h),$$

whence, observing that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$\psi(h) = -\sqrt{-1}\sqrt{\frac{2g}{\pi}}\,.\,h^{-\frac{3}{2}}\,d^{\frac{1}{2}}\{\sqrt{h}\,f(h)\}\left(d.\frac{1}{h}\right)^{-\frac{1}{2}}.$$

By changing h into x, $\psi(x)$ is known, and thus we have a differential equation to the curve.

Suppose, as a particular case, that $f(h) = ch^n$; then, after changing h into $\frac{1}{z}$ for convenience of operation, we have

$$\psi\left(\frac{1}{z}\right) = -\sqrt{-1}\,c\sqrt{\frac{2g}{\pi}}\,.\,z^{\frac{3}{2}}\left(\frac{d}{dz}\right)^{\frac{1}{2}} z^{-n-\frac{1}{2}}$$

$$= c\sqrt{\frac{2g}{\pi}}\,.\,\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\,z^{-n+\frac{1}{2}}$$

by (F), if n be positive. Therefore the equation to the curve is

$$\frac{ds}{dx} = c\sqrt{\frac{2g}{\pi}}\,.\,\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\,x^{n-\frac{1}{2}}.$$

If $n = 0$, or the time be independent of the height, we have

$$\frac{ds}{dx} = c\sqrt{\frac{2g}{\pi}}\,x^{-\frac{1}{2}},$$

which belongs to a cycloid.

If $n = \frac{1}{2}$, or the time vary as the square root of the height,

$$\frac{ds}{dx} = c\sqrt{\frac{g}{2}},$$

which belongs to a straight line.

If $n = 1$, or the time vary as the height,

$$\frac{ds}{dx} = c\,\frac{2\sqrt{2g}}{\pi}\,x^{\frac{1}{2}}:$$

putting $c\,\dfrac{2\sqrt{2g}}{\pi} = \dfrac{1}{\sqrt{a}}$, and solving the equation, we obtain

$$y = \frac{2}{3}\,\frac{(x-a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

We might give many more instances of the use of these formulæ, but our limits will not allow us: and therefore we recommend to such of our readers as are particularly interested in this subject, the original Memoirs above referred to, which are more complete in the applications of the Calculus than in its principles.

ff.

VII.—ON THE TANGENT PLANE OF THE ELLIPSOID.*

In an article, which appeared in the first Number of this Journal, some examples were given of the advantage of employing, in certain cases, a form of the equation to the tangent of the ellipse, which does not involve the coordinates of the point of contact. There is an analogous form of the equation to the tangent plane to the ellipsoid, viz.

$$lx + my + nz = \sqrt{l^2a^2 + m^2b^2 + n^2c^2},$$

l, m, n being the cosines of the angles which a perpendicular on the tangent plane makes with the axes. We shall not give a proof of this, but refer the reader to Hymers' *Geometry of Three Dimensions*, Art. 29. A few examples of its application may perhaps be useful.

1. To find the locus of the intersection of three tangent planes to an ellipsoid, which are mutually at right angles.

Let (lmn), $(l'm'n')$, $(l''m''n'')$ be the cosines of the angles which

* From a Correspondent.

perpendiculars on the planes make with the axes. Their equations will be

$$lx + my + nz = \sqrt{l^2a^2 + m^2b^2 + n^2c^2},$$

$$l'x + m'y + n'z = \sqrt{l'^2a^2 + m'^2b^2 + n'^2c^2},$$

$$l''x + m''y + n''z = \sqrt{l''^2a^2 + m''^2b^2 + n''^2c^2}.$$

Squaring both sides of each of these equations, adding, and observing that, since the planes are at right angles,

$$l^2+l'^2+l''^2=1, \quad m^2+m'^2+m''^2=1, \quad n^2+n'^2+n''^2=1,$$

$$lm+l'm'+l''m''=0, \quad ln+l'n'+l''n''=0, \quad mn+m'n'+m''n''=0,$$

we have

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2.$$

This proposition was first proved by Monge.

2. To find the locus of intersection of the perpendicular on the tangent plane with that plane.

The equation to the tangent plane being

$$lx + my + nz = \sqrt{l^2a^2 + m^2b^2 + n^2c^2} \quad \ldots\ldots (1),$$

the equations to the perpendicular will be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

Multiplying the successive terms on both sides by these equal quantities respectively, we have

$$x^2 + y^2 + z^2 = \sqrt{a^2x^2 + b^2y^2 + c^2z^2},$$

$$\text{or } (x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2.$$

3. Three planes, mutually at right angles, touch three spheres whose radii are a, a', a'', respectively. To find the locus of intersection of the planes.

In this case the equations will be

$$lx + my + nz = a,$$

$$l'x + m'y + n'z = a',$$

$$l''x + m''y + n''z = a''.$$

Squaring, and adding, we shall have

$$x^2 + y^2 + z^2 = a^2 + a'^2 + a''^2.$$

4. Three planes, mutually at right angles, touch three concentric ellipsoids, whose principal sections have their foci coincident. To find the locus of intersection of these planes.

Let abc, $a'b'c'$, $a''b''c''$, be the semi-axes of the three ellipsoids; then the equations will be

$$lx + my + nz = \sqrt{l^2a^2 + m^2b^2 + n^2c^2},$$
$$l'x + m'y + n'z = \sqrt{l'^2a'^2 + m'^2b'^2 + n'^2c'^2},$$
$$l''x + m''y + n''z = \sqrt{l''^2a''^2 + m''^2b''^2 + n''^2c''^2}.$$

Squaring both sides, and adding, we have on the first side $x^2 + y^2 + z^2$, as before. Also,

$$l^2a^2 + m^2b^2 + n^2c^2 = a^2 + m^2(b^2 - a^2) + n^2(c^2 - a^2),$$
$$l'^2a'^2 + m'^2b'^2 + n'^2c'^2 = b'^2 + l'^2(a'^2 - b'^2) + n'^2(c'^2 - b'^2),$$

which, since the foci in the principal sections coincide,

$$= b'^2 + l'^2(a^2 - b^2) + n'^2(c^2 - b^2).$$

Similarly,

$$l''^2a''^2 + m''^2b''^2 + n''^2c''^2 = c''^2 + l''^2(a^2 - c^2) + m''^2(b^2 - c^2).$$

Therefore the second side becomes, by addition,

$$\begin{aligned} &= a^2 + b'^2 + c''^2 \\ &+ a^2(l'^2 + l''^2 - m^2 - n^2) \\ &+ b^2(m^2 + m''^2 - l'^2 - n'^2) \\ &+ c^2(n^2 + n'^2 - l''^2 - m''^2). \end{aligned}$$

But $l^2 + l'^2 + l''^2 = 1 = l^2 + m^2 + n^2$; $\therefore l'^2 + l''^2 = m^2 + n^2$. Similarly,

$$m^2 + m''^2 = l'^2 + n'^2 \text{ and } n^2 + n'^2 = l''^2 + m''^2.$$

Hence the equation to the locus is

$$x^2 + y^2 + z^2 = a^2 + b'^2 + c''^2.$$

Again, since

$$\begin{aligned} a^2 - b^2 &= a'^2 - b'^2, \\ \text{or } a^2 + b'^2 &= a'^2 + b^2, \\ \text{and } b^2 + c''^2 &= b''^2 + c^2; \end{aligned}$$

therefore, by addition,

$$a^2 + b'^2 + c''^2 = a'^2 + b''^2 + c^2.$$

And, in the same way,

$$a^2 + b'^2 + c''^2 = a''^2 + b^2 + c'^2.$$

Consequently the equation may be put into the form

$$x^2 + y^2 + z^2 = \tfrac{1}{3}\{(a^2 + b^2 + c^2) + (a'^2 + b'^2 + c'^2) + (a''^2 + b''^2 + c''^2)\}.$$

This result was first published in the 19th volume of the *Annales de Mathématiques.*

If the equation to the surface be

$$x = \frac{y^2}{a} + \frac{z^2}{a'}.$$

the equation to the tangent plane may be exhibited in the form

$$l(lx + my + nz) + \tfrac{1}{4}(am^2 + a'n^2) = 0.$$

As there is no proof of this in Hymers' Analytical Geometry, we subjoin one.

Let the equation to the tangent plane be

$$lx + my + nz = \delta.$$

This must be identical with the common equation to the tangent plane drawn at a point $(x'y'z')$, viz.

$$x + x' = 2\left(\frac{y'y}{a} + \frac{z'z}{a'}\right),$$

$$\text{or } \frac{2y'y}{ax'} + \frac{2z'z}{a'x'} - \frac{x}{x'} = 1;$$

therefore we must have

$$\frac{l}{\delta} = -\frac{1}{x'}, \text{ or } x' = -\frac{\delta}{l},$$

$$\frac{m}{\delta} = \frac{2y'}{ax'} = -\frac{2ly'}{a\delta},$$

$$\text{whence } y' = -\frac{ma}{2l}.$$

$$\text{Similarly, } z' = -\frac{na'}{2l}.$$

$$\text{But } x' = \frac{y'^2}{a} + \frac{z'^2}{a'};$$

$$\text{hence } -\frac{\delta}{l} = \frac{m^2a}{4l^2} + \frac{n^2a'}{4l^2}.$$

$$\text{or } \delta = -\frac{1}{4l}(m^2a + n^2a').$$

Consequently the equation is

$$l(lx + my + nz) + \tfrac{1}{4}(m^2a + n^2a') = 0.$$

The reader will find no difficulty in proving, by means of this equation, that tangent planes, mutually at right angles, will always intersect in a plane whose equation is

$$x + \tfrac{1}{4}(a + a') = 0;$$

and that the locus of intersection of the tangent plane and the perpendicular upon it from the vertex, is a surface, represented by the equation

$$x(x^2 + y^2 + z^2) + \tfrac{1}{4}(ay^2 + a'z^2) = 0.$$

Q.

VIII.—TRANSFORMATION FROM RECTANGULAR TO POLAR COORDINATES IN DIFFERENTIAL EXPRESSIONS.

When we have a differential expression involving two independent variables, which we wish to transform into one with two other independent variables, the method to be pursued is not at first sight obvious. If, for instance, we have the double integral

$$\iint V\,dx\,dy$$

where V is a function of x and y, and we wish to transform it so as to involve r and θ, the four quantities x, y, r, θ, being connected by the equations

$$x = f(r, \theta),\ y = F(r, \theta),$$

so that

$$dx = \frac{dx}{dr}dr + \frac{dx}{d\theta}d\theta,\ dy = \frac{dy}{dr}dr + \frac{dy}{d\theta}d\theta,$$

we cannot multiply the expressions for dx and dy together as they stand; because in the integral y is supposed to be constant when x varies, and x to be constant when y varies. We must introduce one of these conditions by supposing $dx = 0$ when y varies, or $dy = 0$ when x varies.

Taking the first of these, we have the equations

$$0 = \frac{dx}{dr}dr + \frac{dx}{d\theta}d\theta$$

$$dy = \frac{dy}{dr}dr + \frac{dy}{d\theta}d\theta.$$

Eliminating $d\theta$ between these, we find

$$\frac{dx}{d\theta}dy = \left(\frac{dx}{d\theta}\frac{dy}{dr} - \frac{dx}{dr}\frac{dy}{d\theta}\right)dr.$$

From this it follows, that when $dy = 0$, $dr = 0$. Hence

$$dx = \frac{dx}{d\theta}d\theta.$$

Substituting these values in the double integral, it becomes

$$\iint V\left(\frac{dx}{d\theta}\frac{dy}{dr} - \frac{dx}{dr}\frac{dy}{d\theta}\right)dr\,d\theta,$$

where V involves only r and θ.

When x and y are the rectangular, r, θ the polar coordinates of a point, $x = r\sin\theta,\ y = r\cos\theta,$

and $\iint V\,dx\,dy = \iint V\,rdr\,d\theta.$

If there be three independent variables the same method is to be pursued; but as we have to eliminate between three equations it becomes very complicated, even though we avail ourselves of the method of elimination by cross multiplication given in our first

Number. In the particular case of the quantities representing the coordinates in space of a point, the assumption of a subsidiary quantity much facilitates the calculation. For let

$$\iiint V\, dx\, dy\, dz$$

be the function to be transformed, and

$$x = r\cos\theta,\ y = r\sin\theta\sin\phi,\ z = r\sin\theta\cos\phi.$$

Assume $\rho = r\sin\theta$. Then we have

$$y = \rho\sin\phi,\quad z = \rho\cos\phi \ldots\ldots\ldots(1),$$

$$\rho = r\sin\theta,\quad x = r\cos\theta \ldots\ldots\ldots(2).$$

We can then transform, as we did before, from x, y, z, to x, ρ, ϕ, and after that from x, ρ, ϕ to r, θ, ϕ.

The simplicity of the method consists in this, that the second operation is exactly similar to the first, so that we do not require to repeat the calculation, but merely to write down the result, substituting r for ρ, and θ for ϕ. Effecting the first operation,

$$dx\, dy\, dz = dx\, d\rho\, d\phi \left\{\frac{dy}{d\phi}\frac{dz}{d\rho} - \frac{dy}{d\rho}\frac{dz}{d\phi}\right\}$$

which by equations (1) becomes

$$dx\, dy\, dz = \rho\, dx\, d\rho\, d\phi.$$

Again, changing from x, ρ, ϕ to r, θ, ϕ, we find

$$dx\, dy\, dz = \rho r\, dr\, d\theta\, d\phi = r^2\, dr \sin\theta\, d\theta\, d\phi.$$

Therefore

$$\iiint V\, dx\, dy\, dz = \iiint V r^2\, dr \sin\theta\, d\theta\, d\phi.$$

The same assumption may be usefully applied to transforming the expression

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}$$

from rectangular to polar coordinates; for it will be seen, that having found $\dfrac{dV}{dy}$ and $\dfrac{d^2V}{dy^2}$, the whole expression may be written down without further trouble.

Thus we find

$$\frac{dV}{dy} = \sin\phi\frac{dV}{d\rho} + \frac{\cos\phi}{\rho}\frac{dV}{d\phi},$$

$$\frac{d^2V}{dy^2} = \sin^2\phi\frac{d^2V}{d\rho^2} + \frac{\cos^2\phi}{\rho^2}\frac{d^2V}{d\phi^2} + \frac{\cos^2\phi}{\rho}\frac{dV}{d\rho}$$

$$+ \frac{2\sin\phi\cos\phi}{\rho^2}\left(\rho\frac{d^2V}{d\rho\, d\phi} - \frac{dV}{d\phi}\right).$$

The expression for $\dfrac{d^2V}{dz^2}$ is got by putting $90 - \phi$ for ϕ in the above, so that without writing it down

$$\frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = \frac{d^2V}{d\rho^2} + \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} + \frac{1}{\rho}\frac{dV}{d\rho}.$$

Putting r for ρ and θ for ϕ, we have the similar expression

$$\frac{d^2V}{d\rho^2}+\frac{d^2V}{dx^2}=\frac{d^2V}{dr^2}+\frac{1}{r^2}\frac{d^2V}{d\theta^2}+\frac{1}{r}\frac{dV}{dr}.$$

And the expression for $\frac{dV}{d\rho}$ in r and θ being similar to that for $\frac{dV}{dy}$ in r and ϕ, gives

$$\frac{1}{\rho}\frac{dV}{d\rho}=\frac{1}{r}\frac{dV}{dr}+\frac{\cot\theta}{r^2}\frac{dV}{d\theta}.$$

Adding these three expressions,

$$\begin{aligned}\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}&=\frac{1}{r^2}\left(\frac{d^2V}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{dV}{d\theta}\right)\\&\qquad+\frac{1}{r^2\sin^2\theta}\frac{d^2V}{d\phi^2}+\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}\\&=\frac{1}{r^2}\frac{d}{d\cos\theta}\left(\sin^2\theta\frac{dV}{d\cos\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{d^2V}{d\phi^2}+\frac{1}{r}\frac{d^2(rV)}{dr^2},\end{aligned}$$

which is the well known expression.

A. S.

IX.—ON THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS.

The integration of Partial Differential Equations is much facilitated by the principle which we have developed in our preceding Numbers, and it is the more remarkable, that it has been so little applied to these equations, as the first step was taken many years ago. Fourier, in his *Traité de Chaleur*, published in 1822, has shown that the series which are obtained in the solution of several partial differential equations may be conveniently expressed by the separation of the symbols of operation from those of quantity. But though he has used this method very frequently, yet he appears to have had some unwillingness to give himself up to it entirely as a guide in his investigations, as if he were not familiar with the principles on which it is founded. His idea apparently was, that the expression which he obtained as solutions might be conveniently expressed by separating the symbols of operation, and not that the symbolical expressions are the proper solutions of the equations, and the series merely the expansion of them. Other French writers seem to have avoided carefully entering at all on the track

which Fourier opened: Poisson in particular, in a digression on the subject of partial differential equations in the second volume of his *Mécanique*, does not put in the symbolical form the solution of a very simple equation, which is so given by Fourier. Mr. Greatheed, in a paper published in the number of the Philosophical Magazine for September 1837, was, we believe, the first to call the attention of mathematicians to the utility of this method in the case of partial differential equations, but he had not then reduced it to its greatest degree of simplicity; and his paper is chiefly occupied with a particular class of equations of the first order with variable coefficients, which are not so interesting as many others with constant coefficients. We shall here, therefore, proceed to give several examples of the application of the principles which we laid down in Art. V. of our first Number.

And, first, we may observe generally, that linear partial differential equations between any number of variables with constant coefficients, are to be treated exactly like ordinary differential equations with regard to one of the variables, the symbols of operation of the others being treated as constants. If, for instance, we have the equation

$$a\frac{dz}{dx} + b\frac{dz}{dy} = c.$$

This may be put under the form

$$\left(a\frac{d}{dx} + b\frac{d}{dy}\right)z = c,$$

and therefore

$$z = \left(a\frac{d}{dx} + b\frac{d}{dy}\right)^{-1}(c + 0),$$

where we suppose x to be the variable, and $\dfrac{d}{dy}$ a constant with regard to it. Now the operation $\left(a\dfrac{d}{dx} + b\dfrac{d}{dy}\right)^{-1}$ by the theorem given in page 25, is equivalent to

$$a^{-1}\,\varepsilon^{-\frac{b}{a}x\frac{d}{dy}}\int dx\,\varepsilon^{\frac{b}{a}x\frac{d}{dy}},$$

which being performed, gives

$$z = \left(\frac{d}{dy}\right)^{-1}\frac{c}{b} + \frac{1}{a}\,\varepsilon^{-\frac{b}{a}x\frac{d}{dy}}f(y),$$

$f(y)$ being an arbitrary function of y, taking the place of the constant in ordinary differential equations.

From this expression we get

$$z = \frac{cy}{b} + f(ay - bx),$$

as by Taylor's theorem

$$\varepsilon^{h\frac{d}{dx}} f(x) = f(x + h).$$

We might equally well have supposed y to be the variable, and $\dfrac{d}{dx}$ a constant with regard to it.

Again, taking the well-known equation for the motion of waves,

$$\frac{d^2 z}{dt^2} - a^2 \frac{d^2 z}{dx^2} = 0,$$

it may be put under the form

$$\left(\frac{d^2}{dt^2} - a^2 \frac{d^2}{dx^2}\right) z = 0;$$

and integrating it like the ordinary differential equation

$$\left(\frac{d^2}{dt^2} - n^2\right) z = 0,$$

$$\text{we find } z = \varepsilon^{at\frac{d}{dx}} \phi(x) + \varepsilon^{-at\frac{d}{dx}} \psi(x),$$

$$\text{or } z = \phi(x + at) + \psi(x - at).$$

The equation $r - 2a\, s + a^2 t = 0$ may be put under the form

$$\left(\frac{d}{dx} - a\frac{d}{dy}\right)^2 z = 0,$$

the solution of which is by the Theorem in page 25,

$$z = \varepsilon^{ax\frac{d}{dy}} \textstyle\int^2 dx^2 . 0$$

$$= \varepsilon^{ax\frac{d}{dy}} \{x\phi(y) + \psi(y)\},$$

$\phi(y)$ and $\psi(y)$ being arbitrary functions of y arising from the integration. Hence, finally, we have

$$z = x\phi(y + ax) + \psi(y + ax).$$

The equation

$$r - a^2 t + 2abp + 2a^2 bq = 0$$

is equivalent to

$$\left\{\frac{d}{dx} - \left(a\frac{d}{dy} - 2ab\right)\right\}\left(\frac{d}{dx} + a\frac{d}{dy}\right) z = 0.$$

Integrating with regard to the first factor, we have

$$\left(\frac{d}{dx} + a\frac{d}{dy}\right) z = \varepsilon^{x\left(a\frac{d}{dy} - 2ab\right)} \phi(y).$$

Integrating with regard to the remaining factor

$$z = \varepsilon^{-ax\frac{d}{dy}} \int dx\, \varepsilon^{2ax\left(\frac{d}{dy}-b\right)} \phi(y) + \varepsilon^{-ax\frac{d}{dy}} \psi(y)$$

$$= \varepsilon^{-ax\frac{d}{dy}} \varepsilon^{2ax\left(\frac{d}{dy}-b\right)} \left\{2a\left(\frac{d}{dy} - b\right)\right\}^{-1} \phi(y) + \psi(y - ax)$$

$$= \varepsilon^{ax\frac{d}{dy}} \varepsilon^{-2abx} \varepsilon^{by} \phi_1(y) + \psi(y - ax)$$

by changing the arbitrary function

$$= \varepsilon^{-2abx} \varepsilon^{b(y+ax)} \phi_1(y + ax) + \psi(y - ax)$$

$$= \varepsilon^{b(y-ax)} \phi_1(y + ax) + \psi(y - ax).$$

Let us take also the equation

$$r - c^2 t = xy.$$

Without operating on each factor separately, we may arrive more readily at the result by the same means as those employed in page 28 of our first Number. For we have

$$z = \left(\frac{d^2}{dx^2} - c^2 \frac{d^2}{dy^2}\right)^{-1} (xy) + \phi(y + ax) + \psi(y - ax)$$

$$= \frac{d^{-2}}{dx^{-2}} \left(1 - c^2 \frac{d^2}{dy^2} \frac{d^{-2}}{dx^{-2}}\right)^{-1} (xy) + \phi(y + ax) + \psi(y - ax)$$

$$= \left(1 - c^2 \frac{d^2}{dy^2} \frac{d^{-2}}{dx^{-2}}\right)^{-1} \left(\frac{x^3 y}{6}\right) + \phi(y + ax) + \psi(y - ax).$$

Therefore

$$z = \frac{x^3 y}{6} + \phi(y + ax) + \psi(y - ax),$$

the arbitrary functions being added as in the second example.

A very simple equation being one which occurs in the theory of heat is

$$\frac{dv}{dt} = a\frac{d^2 v}{dx^2},$$

which is the expression for the rectilinear propagation of heat. The solution of this is easily seen to be, if we integrate with regard to t,

$$v = \varepsilon^{at\frac{d^2}{dx^2}} f(x),$$

$f(x)$ being an arbitrary function. If the sign of operation be expanded, we shall obtain a series which is the solution derived by Poisson from the method of indeterminate coefficients. Laplace has deduced from the series an elegant expression for the solution under the form of a definite integral; but it may be more easily deduced from the symbolical solution in the following manner.

Since $$\int_{-\infty}^{\infty} \varepsilon^{-\omega^2}\, d\omega = \sqrt{\pi},$$

and also $$\int_{-\infty}^{\infty} \varepsilon^{-(\omega+b)^2}\, d\omega = \sqrt{\pi},$$

we can put the expression for v under the form

$$\sqrt{\pi}\, v = \int_{-\infty}^{\infty} d\omega\, \varepsilon^{at\frac{d^2}{dx^2}}\, \varepsilon^{-\left(\omega - \sqrt{at}\frac{d}{dx}\right)^2} f(x)$$

$$= \int_{-\infty}^{\infty} d\omega\, \varepsilon^{-\omega^2} \cdot \varepsilon^{2\omega\sqrt{at}\frac{d}{dx}} f(x)$$

$$= \int_{-\infty}^{\infty} d\omega\, \varepsilon^{-\omega^2} f(x + 2\omega\sqrt{at})$$

by Taylor's theorem; and this is Laplace's expression.

This equation $\dfrac{dv}{dt} = a\dfrac{d^2v}{dx^2}$ not being homogeneous in the index of the operations, admits of two solutions of very different characters. The one, which we have found by integrating with regard to t, contains only one arbitrary function of x. The other, which may be found by integration with regard to x, must contain two arbitrary functions of t, as the index of operation is of the second degree. If we write the equation in the form

$$\frac{d^2v}{dx^2} - \frac{1}{a}\frac{dv}{dt} = 0,$$

and integrate by the method employed in page 28, we find for the integral

$$v = \left(x + \frac{1}{a}\frac{x^3}{1.2.3}\frac{d}{dt} + \frac{1}{a^2}\frac{x^5}{1.2.3.4.5}\frac{d^2}{dt^2} + \&c.\right)\phi(t)$$

$$+ \left(1 + \frac{1}{a}\frac{x^2}{1.2}\frac{d}{dt} + \frac{1}{a^2}\frac{x^4}{1.2.3.4}\frac{d^2}{dt^2} + \&c.\right)\psi(t).$$

It seems at first sight anomalous, that the same equation should have two solutions so different in character: the following is the explanation of the difficulty. Since by Maclaurin's theorem any function of a variable may be expressed by means of its differential coefficients taken with regard to that variable, for the particular value 0 of the variable, we know the function if we can determine its successive differential coefficients. Now from the equation $\dfrac{dv}{dt} = a\dfrac{d^2v}{dx^2}$ we can, when we know the value of v when $t=0$, determine the values of all the differential coefficients with regard to t when $t = 0$. So that in the resulting expression deduced from Maclaurin's theorem there is only one quantity left undetermined, which is the arbitrary function $f(x)$, introduced in the integration. But from the equation $\dfrac{d^2v}{dx^2} = \dfrac{1}{a}\dfrac{dv}{dt}$, we can only determine from the value of v when $x = 0$ the values of the alternate differential coefficients with regard to x. There must consequently be introduced another undetermined quantity, namely, the value of $\dfrac{dv}{dx}$ when $x = 0$, for knowing these two quantities we can

determine the values of all the successive differential coefficients with regard to x.

The equation for determining the vibratory motion of an elastic spring is

$$\frac{d^2v}{dt^2}+\frac{d^4v}{dx^4}=0.$$

The solution of which is readily seen to be

$$v=\cos\left(t\frac{d^2}{dx^2}\right)\mathrm{F}(x)+\sin\left(t\frac{d^2}{dx^2}\right)f(x).$$

The equation for determining the vibratory motion of elastic plates is

$$\frac{d^2v}{dt^2}+\frac{d^4v}{dx^4}+2\frac{d^4v}{dx^2\,dy^2}+\frac{d^4v}{dy^4}=0,$$

which may be put under the form

$$\frac{d^2v}{dt^2}+\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)^2v=0;$$

the integral of which is

$$v=\cos t\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\mathrm{F}(x,y)+\sin t\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)f(x,y).$$

In investigating the motion of heat in a ring, we obtain as an equation for determining the temperature at any time and point,

$$\frac{dv}{dt}=k\frac{d^2v}{dx^2}-h\,v;$$

the solution of which is

$$v=\varepsilon^{\left(k\frac{d^2}{dx^2}-h\right)t}f(x)$$
$$=\varepsilon^{-ht}\,\varepsilon^{kt\frac{d^2}{dx^2}}f(x),$$

an expression closely connected with one previously given.

In the examples which we have given, the coefficients are constant, but if one of the variables only enters into the coefficients, and we integrate with regard to the other, as the one variable is unaffected by the sign of operation with regard to the other, it may be considered as a constant in the integration.

A good example of this kind of equation is that which expresses the motion of heat in a solid cylinder of infinite length, namely

$$\frac{dv}{dt}=a\left(\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}\right),$$

the solution of which is

$$v=\varepsilon^{at\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}\right)}f(x),$$

or, as it may be expressed,

$$v = \varepsilon^{at\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right)} f(x).$$

But this will only be possible when the integration is of the first degree, since, as we showed in our first Number, the symbols of operation become subject to different laws when the variable itself and the sign of differentiation are both involved. And this leads us to the consideration of equations with variable coefficients. As in the case of ordinary differential equations, the solution of this class is attended with great difficulties, so as to become almost impossible for equations of an order higher than the first, it is to be supposed that these difficulties are no way diminished in the case of partial differential equations. Of these, however, when of the first order, Mr. Greatheed has shown, that a large class may be solved like ordinary differential equations. It is included under the form

$$\frac{dz}{dx} + \mathrm{XY}\frac{dz}{dy} = \mathrm{P}z + \mathrm{Q},$$

where X is a function of x only, Y of y only, and P and Q of both x and y. This may be reduced to the form

$$\frac{dz}{dx} + \mathrm{X}\frac{dz}{dy} = \mathrm{P}z + \mathrm{Q},$$

by a change of the variable; for if $y' = \int\frac{dy}{\mathrm{Y}}$, then $\mathrm{Y}\frac{dz}{dy} = \frac{dz}{dy'}$: so that it is only necessary to consider the latter equation. Let it be put under the form

$$\frac{dz}{dx} + \left(\mathrm{X}\frac{d}{dy} - \mathrm{P}\right)z = \mathrm{Q},$$

and treated as an ordinary linear differential equation between z and x. If we integrate by the method of integrating factors, the factor is $\varepsilon^{\int dx\left(\mathrm{X}\frac{d}{dy} - \mathrm{P}\right)}$, and the equation becomes

$$\frac{d}{dx}\left\{\varepsilon^{\int dx\left(\mathrm{X}\frac{d}{dy} - \mathrm{P}\right)}.z\right\} = \varepsilon^{\int dx\left(\mathrm{X}\frac{d}{dy} - \mathrm{P}\right)}.\mathrm{Q};$$

whence

$$z = \varepsilon^{-\int dx\left(\mathrm{X}\frac{d}{dy} - \mathrm{P}\right)}\int dx\left\{\varepsilon^{\int dx\left(\mathrm{X}\frac{d}{dy} - \mathrm{P}\right)}.\mathrm{Q}\right\} + \varepsilon^{-\int dx\left(\mathrm{X}\frac{d}{dy} - \mathrm{P}\right)}.\phi(y),$$

$\phi(y)$ being an arbitrary function of y.

We shall take as an example the equation

$$x\frac{dz}{dx} + y\frac{dz}{dy} = nz.$$

Let $\frac{dy}{y} = dt$, and therefore $y = \varepsilon^t$. Then

$$\frac{dz}{dx} + \frac{1}{x}\frac{dz}{dt} = n\frac{z}{x}.$$

The integral of which, by the preceding formula, is

$$z = \varepsilon^{-\int dx \left(\frac{1}{x}\frac{d}{dt} - \frac{n}{x}\right)} \phi(t),$$

$$\text{or } z = \varepsilon^{n \log x} \cdot \varepsilon^{-\log x \frac{d}{dt}} \phi(t)$$

$$= x^n \phi(t - \log x) = x^n \phi(\log y - \log x) = x^n \psi\left(\frac{y}{x}\right).$$

The equation $\qquad x\dfrac{dz}{dx} - y\dfrac{dz}{dy} = \dfrac{x^2}{y}$

may be integrated in the same way. Or if we change both the independent variables, making $\dfrac{dy}{y} = dt$ and $\dfrac{dx}{x} = du$, it becomes

$$\left(\frac{d}{du} - \frac{d}{dt}\right) z = \varepsilon^{2u} \varepsilon^{-t},$$

which gives $z = \varepsilon^{u\frac{d}{dt}} \int du\, \varepsilon^{-u\frac{d}{dt}} \varepsilon^{2u} \varepsilon^{-t} + \varepsilon^{u\frac{d}{dt}} \phi(t)$

$$= \varepsilon^{u\frac{d}{dt}} \int du\, \varepsilon^{3u} \varepsilon^{-t} + \varepsilon^{u\frac{d}{dt}} \phi(t)$$

$$= \frac{\varepsilon^{2u} \varepsilon^{-t}}{3} + \phi(t + u)$$

$$= \frac{x^2 y}{3} + \psi(xy),$$

as $u = \log x$, $t = \log y$.

The equation $\qquad y\dfrac{dz}{dx} + x\dfrac{dz}{dy} = z$

may, by changing $y\,dy$ into $\frac{1}{2}\,d.y^2$, be transformed into

$$\frac{dz}{dx} + \left(2x\frac{d}{d.y^2} - \frac{1}{y}\right) z = 0.$$

The integral of which is

$$z = \varepsilon^{-\int dx \left(2x \frac{d}{d.y^2} - \frac{1}{y}\right)} \phi(y^2)$$

$$= \varepsilon^{\int \frac{dx}{y}} \varepsilon^{-x^2 \frac{d}{d.y^2}} \phi(y^2)$$

$$= \varepsilon^{-x^2 \frac{d}{d.y^2}} \left\{\varepsilon^{x^2 \frac{d}{d.y^2}} \left(\varepsilon^{\int \frac{dx}{y}}\right) \phi(y^2)\right\}$$

$$= \varepsilon^{-x^2 \frac{d}{d.y^2}} \left\{\varepsilon^{\int \frac{dx}{\sqrt{y^2+x^2}}} \phi(y^2)\right\}$$

$$= \varepsilon^{-x^2 \frac{d}{d.y^2}} \left\{\varepsilon^{\log(x + \sqrt{x^2+y^2})} \cdot \phi(y^2)\right\}$$

$$= \varepsilon^{-x^2 \frac{d}{d.y^2}} \left\{(x + \sqrt{x^2 + y^2})\, \phi(y^2)\right\}$$

$$= (x + y)\, \phi(y^2 - x^2).$$

Other equations, which at first sight do not appear to come under this form, may be reduced to it by a proper assumption of a new independent variable. For instance, the equation

$$(x+y)\frac{dz}{dx} + (y-x)\frac{dz}{dy} = z,$$

is converted into

$$u\frac{dz}{dx} - 2x\frac{dz}{du} = z,$$

by supposing $y = u - x$. These however are particular cases which come under no general rule.

D. F. G.

X.—ON STABLE AND UNSTABLE EQUILIBRIUM.*

Let P, P′, P″...... be any number of forces acting upon a material system, and let dp, dp', dp''...... be the virtual velocities of their points of application, estimated positive in the directions in which the forces act; then, when P, P′, P″...... are in equilibrium, the equilibrium is *stable* when $\int(\mathrm{P}dp + \mathrm{P}'dp' + \ldots\ldots)$ is a *maximum;* and *unstable* when this is *not a maximum:* and *vice versâ.*

We shall demonstrate this in the case of one rigid body; and the same reasoning can easily be applied when the system contains several rigid bodies.

When P, P′, P″...... act upon a rigid body, they can always be reduced to two forces, but not to one. Let R and R′ be these forces, and dr, dr' the virtual velocities of their points of application: then

$$\mathrm{P}dp + \mathrm{P}'dp' + \ldots\ldots = \mathrm{R}dr + \mathrm{R}'dr',$$

$$\therefore u = \int(\mathrm{P}dp + \mathrm{P}'dp' + \ldots\ldots) = \int(\mathrm{R}dr + \mathrm{R}'dr').$$

Hence $$du = \mathrm{R}dr + \mathrm{R}'dr',$$

and $$d^2u = \mathrm{R}d^2r + d\mathrm{R}dr + \mathrm{R}'d^2r' + d\mathrm{R}'dr'.$$

Now when P, P′, P″...... are in equilibrium,

$$\mathrm{P}dp + \mathrm{P}'dp' + \mathrm{P}''dp'' + \ldots\ldots = 0,$$

$$\therefore \mathrm{R}dr + \mathrm{R}'dr' = 0, \quad \therefore du = 0;$$

and, since dr and dr' are independent of each other,

$$\mathrm{R} = 0 \text{ and } \mathrm{R}' = 0,$$

$$\therefore d^2u = d\mathrm{R}.dr + d\mathrm{R}'.dr'.$$

When u is a maximum, d^2u is negative, and therefore (since dr and dr' are independent of each other) $d\mathrm{R}.dr$ and $d\mathrm{R}'.dr'$ are both

* From a Correspondent.

negative. Hence if dr be positive, dR is negative; and *vice versâ:* and so with dr' and dR'.

From this we learn, that if the body receive any slight displacement from the situation of equilibrium, two small forces dR and dR' are brought into play, which act opposite to the directions in which the points, at which they act, have moved in consequence of the displacement of the body. Hence the effect of these forces is to draw the body back towards its position of equilibrium: and therefore the equilibrium was *stable.*

Conversely, in order that the equilibrium may be stable, the forces dR, dR' put in play by the displacement must act so as to carry the body back to its situation of equilibrium: and therefore the virtual velocities dr and dr' must have different signs to dR and dR' respectively: hence d^2u is negative, and therefore u a maximum.

If u be a *minimum* we may shew in the same manner that the equilibrium is *unstable,* and *vice versâ.*

If u be *neither a maximum nor a minimum,* then d^2u (or d^3u, if d^2u vanishes) will be positive for some displacements, and negative for others; and will therefore be apparently *dubious,* though in fact it will be *unstable;* for although a displacement may be found which shall cause the body to return to its position of rest, yet in oscillating through the position of rest it may come to one, from which it will not return to its position of rest. The converse of this is also true.

Hence, if by *stable* equilibrium we mean, that for all small displacements the body oscillates about its position of rest, then the equilibrium is *stable* when u is a *maximum,* and in no other case: in all other cases the equilibrium is *unstable.*

N.B. We have supposed that $Pdp + P'dp' + \dots$ is a perfect differential. If this be not the case, the above reasoning is still true; but the result should be stated thus: when the equilibrium is *stable,* the first differential of $Pdp + P'dp' + \dots$ (or if that and the second vanish, the third) is *negative,* and *vice versa:* in all other cases the equilibrium is *unstable.*

If the system consist of several bodies, and R, R', R'', R'''...... be the smallest number of forces to which the system of forces acting on the bodies can be reduced, then

$$du = Rdr + R'dr' + R''dr'' + R'''dr''' + \dots$$

in which dr, dr', dr'', dr''' ...... are independent of each other: and the reasoning upon these and the forces R, R', R'', R''' ...... will be as above.

The following is an application of this principle to fluid equilibrium.

Problem. Required the form of equilibrium of a mass of incompressible fluid, every particle of which is attracted towards a centre of force, varying as a function of the distance; and to determine whether the equilibrium be *stable* or *unstable.*

Let the fluid be referred to polar coordinates, the centre of force being the origin, θ the angle which r makes with a fixed line through the origin, and ω the angle which the plane through r and the fixed line makes with a fixed plane passing through the fixed line: and let (as usual) $\cos\theta = \mu$. It is evident that the fluid must completely surround the centre of force: let r and r' be the radii of the external and internal surfaces; these are functions of μ and ω. The volume of the fluid is given: let c be the radius of the sphere which has the same volume: hence

$$\frac{4\pi}{3}c^3 = \int_{-1}^{1}\int_0^{2\pi}\int_{\mathrm{r}'}^{\mathrm{r}} r^2\, d\mu\, d\omega\, dr = \tfrac{1}{3}\int_{-1}^{1}\int_0^{2\pi}(\mathrm{r}^3 - \mathrm{r}'^3)\, d\mu\, d\omega.$$

Since r^3 and r'^3 are functions of μ and ω, they may be expanded in series of Laplace's coefficients: let

$$\mathrm{r}^3 = a^3 + \alpha(\mathrm{Y}_1 + \mathrm{Y}_2 + \ldots\ldots) = a^3 + \alpha y,$$

$$\text{and } \mathrm{r}'^3 = a'^3 + \alpha(\mathrm{Y}_1' + \mathrm{Y}_2' + \ldots\ldots) = a'^3 + \alpha y',$$

where α is a numerical coefficient, the use of which will be seen hereafter.

Then, by a fundamental property of these functions,

$$\int_{-1}^{1}\int_0^{2\pi} y\, d\mu\, d\omega = 0;$$

and similarly of y'. Hence we have, by the above equation,

$$c^3 = a^3 - a'^3.$$

Let $f(r)$ be the force of attraction on a unit of mass at a distance r from the centre of force: then $f(r)\,\delta m$ is the attraction on the element of the mass δm: this attraction acts in the line r, and tends to shorten r: hence, in this case,

$$u = \int \Sigma.\{-f(r)\,\delta m\}\, dr; \text{ and this } = \Sigma.\int\{-f(r)\, dr\}\,\delta m.$$

The symbol of integration Σ refers to the differential of the mass δm. Now δm equals (as above, if density $= 1$)

$$r^2 \sin\theta\, d\theta\, d\omega\, dr = -\, r^2\, d\mu\, d\omega\, dr,$$

and the limits of r, ω, μ are respectively r' and r, 0 and 2π, 1 and -1: then, putting

$$\int -f(r)\, dr = f_1(r), \text{ and } \int r^2 f_1(r)\, dr = \phi(r^3),$$

we have

$$u = \int_{-1}^{1}\int_0^{2\pi}\int_{\mathrm{r}'}^{\mathrm{r}} r^2 f_1(r)\, d\mu\, d\omega\, dr = \int_{-1}^{1}\int_0^{2\pi}\{\phi(\mathrm{r}^3) - \phi(\mathrm{r}'^3)\}\, d\mu\, d\omega.$$

But $\phi(\mathrm{r}^3) - \phi(\mathrm{r}'^3) = \phi(a^3 + \alpha y) - \phi(a'^3 + \alpha y')$

$$= \phi(a^3) - \phi(a'^3) + \alpha y\,\phi'(a^3) - \alpha y'\,\phi'(a'^3)$$
$$+ \tfrac{1}{2}\alpha^2 y^2\,\phi''(a^3) - \tfrac{1}{2}\alpha^2 y'^2\,\phi''(a'^3) + \ldots$$

in which the accents to ϕ denote differentiation with respect to a^3 and a'^3. Hence

$$u = 4\pi\,\phi(a^3) - 4\pi\,\phi(a'^3)$$
$$+ \tfrac{1}{2}\alpha^2\int_{-1}^{1}\int_0^{2\pi}\{y^2\phi''(a^3) - y'^2\phi''(a'^3)\}\, d\mu\, d\omega + \ldots$$

It is very easily shewn, that $\phi''(a^3) = -\frac{1}{9a^2}f(a)$, and is therefore *negative* in every case of *attraction*, (but positive if the force were repulsive).

From these calculations we gather the following results. If α be a very small quantity, the increment of u above what it becomes when $\alpha = 0$ (that is, when $r = a$ and $r' = a'$) does not involve the first power of α: hence, when $r = a$ and $r' = a'$, we have $du = 0$, and therefore the fluid is in equilibrium.

Hence a form of equilibrium is that of a hollow sphere of any dimensions (so long as the *volume* be constant), and having the centres of the bounding surfaces at the centre of force.

And the equilibrium of the *external* surface is *stable*, because for all values of y when $y' = 0$ the multiplier of α^2 is negative, and therefore u is a *maximum:* and the equilibrium of the *internal* surface is *unstable*, because for all values of y' when $y = 0$ the multiplier of α^2 is positive, and therefore u is a *minimum.* On the whole, the equilibrium is dubious.

Cor. 1. If $a' = 0$, then $y' = 0$, and the equilibrium is altogether stable.

Cor. 2. If the force were *repulsive*, and the fluid were held in equilibrium by a rigid spherical envelope; then $y = 0$, and the multiplier of α^2 would be negative, since $\phi''(a'^3)$ is then positive, and u is a *maximum*, and the equilibrium of the internal surface is *stable.*

J. H. P.

XI.—ANALYTICAL GEOMETRY OF THREE DIMENSIONS. No. II.

We proceed to give some instances of the advantage of symmetry in applying the Differential Calculus to Geometry of three dimensions. The following form of the equation to the tangent plane is probably known to many of our readers; but, as Leroy has only deduced it from that in terms of the partial differential coefficients of one of the coordinates with respect to the other two, we will give an independent and easy proof of it.

1. If $F(x, y, z) = 0$, be the equation to a surface, the locus of the tangent lines drawn to it at a point (x, y, z) is a plane whose equation is

$$(x' - x)\frac{dF}{dx} + (y' - y)\frac{dF}{dy} + (z' - z)\frac{dF}{dz} = 0 \ldots\ldots(1).$$

The equation to any line drawn through the point x, y, z, is

$$\frac{x'-x}{l} = \frac{y'-y}{m} = \frac{z'-z}{n} \text{......(2).}$$

This line will in general be cut by the surface in one or more other points. Let the coordinates of the nearest of these be x_1, y_1, z_1, and let each member of equation (2), when x', y', z' become x_1, y_1, z_1, be assumed equal to r, then

$$x_1 = x + lr, \quad y_1 = y + mr, \quad z_1 = z + nr;$$

$$\text{but } \mathrm{F}(x_1, y_1, z_1) = 0;$$

$$\text{therefore } \mathrm{F}\{x + lr,\ y + mr,\ z + nr\} = 0.$$

Expanding and observing that $\mathrm{F}(x, y, z) = 0$,

$$l\frac{d\mathrm{F}}{dx} + m\frac{d\mathrm{F}}{dy} + n\frac{d\mathrm{F}}{dz} + \frac{r}{2}\left(l^2\frac{d^2\mathrm{F}}{dx^2} + \&c.\right) + \text{......} = 0.$$

When the line becomes a tangent, the points (x, y, z), (x_1, y_1, z_1) approach indefinitely near to one another, and r becomes indefinitely small; therefore, taking the limit of the preceding equation,

$$l\frac{d\mathrm{F}}{dx} + m\frac{d\mathrm{F}}{dy} + n\frac{d\mathrm{F}}{dz} = 0.$$

Multiplying the several terms of this by the several members of equation (2), we obtain

$$(x'-x)\frac{d\mathrm{F}}{dx} + (y'-y)\frac{d\mathrm{F}}{dy} + (z'-z)\frac{d\mathrm{F}}{dz} = 0,$$

for the equation to the locus of the tangent lines.

2. Since the normal line is perpendicular to the tangent plane, its equations are

$$\frac{x'-x}{\frac{d\mathrm{F}}{dx}} = \frac{y'-y}{\frac{d\mathrm{F}}{dy}} = \frac{z'-z}{\frac{d\mathrm{F}}{dz}} \text{............ (3).}$$

3. We shall next investigate an expression in terms of the partial differential coefficients of $\mathrm{F}(x, y, z)$, for the radius of curvature of a section of a surface made by any plane passing through the normal at a given point.

Let $\frac{d\mathrm{F}}{dx} = \mathrm{U}$, $\frac{d\mathrm{F}}{dy} = \mathrm{V}$, $\frac{d\mathrm{F}}{dz} = \mathrm{W}$, then the equations to the normal are

$$\frac{x'-x}{\mathrm{U}} = \frac{y'-y}{\mathrm{V}} = \frac{z'-z}{\mathrm{W}} \text{............ (4).}$$

Again, if ds be an element of the section, and dx, dy, dz its projections, $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the cosines of the angles which it

makes with the axes, and therefore the equation to the normal plane to the section is

$$(x' - x)\,dx + (y' - y)\,dy + (z' - z)\,dz = 0 \;\ldots\ldots (5).$$

The line of intersection of two consecutive normal planes will be determined by (5) and its differential, which is

$$(x' - x)\,d^2x + (y' - y)\,d^2y + (z' - z)\,d^2z - ds^2 = 0 \ldots\ldots (6);$$

and the intersection of this line with the normal to the surface will evidently be the centre of curvature of the section. We may suppose x', y', z' to belong to this point; they will then be the same in (4) as in (5) and (6): and assuming each member of (4) equal to Q, and substituting for x', y', z' in (6), we have

$$(\mathrm{U}\,d^2x + \mathrm{V}\,d^2y + \mathrm{W}\,d^2z)\,\mathrm{Q} - ds^2 = 0.$$

But if ρ be the radius of curvature

$$\rho^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2 = (\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2)\,\mathrm{Q}^2,$$

therefore

$$\rho = \pm \frac{\sqrt{\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2}.ds^2}{\mathrm{U}\,d^2x + \mathrm{V}\,d^2y + \mathrm{W}\,d^2z} \;\ldots\ldots\ldots (7).$$

This expression may be transformed as follows. Since

$$\mathrm{U}\,dx + \mathrm{V}\,dy + \mathrm{W}\,dz = 0,$$

differentiating, and assuming $\dfrac{d^2\mathrm{F}}{dx^2} = u$, $\dfrac{d^2\mathrm{F}}{dy^2} = v$, $\dfrac{d^2\mathrm{F}}{dz^2} = w$, $\dfrac{d^2\mathrm{F}}{dy\,dz} = u'$, $\dfrac{d^2\mathrm{F}}{dz\,dx} = v'$, $\dfrac{d^2\mathrm{F}}{dx\,dy} = w'$,

$$\mathrm{U}\,d^2x + \mathrm{V}\,d^2y + \mathrm{W}\,d^2z + u\,dx^2 + v\,dy^2 + w\,dz^2$$
$$+ 2u'\,dy\,dz + 2v'\,dz\,dx + 2w'\,dx\,dy = 0.$$

Employing this equation, and assuming $\dfrac{dx}{ds} = l$, $\dfrac{dy}{ds} = m$, $\dfrac{dz}{ds} = n$, (7) is changed into

$$\rho = \mp \frac{\sqrt{\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2}}{l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw'} \;\ldots\ldots (8).$$

4. It may be as well here to show how formulæ involving the partial differential coefficients of F, may readily be transformed into others in terms of the partial differential coefficients of one of the coordinates as z. Suppose the equation to be put under the form $f(x, y) - z = 0$, so that $\mathrm{F}(x, y, z) = f(x, y) - z$. Then

$$\left.\begin{aligned}
&\frac{d\mathrm{F}}{dx} = \frac{d}{dx}f(x, y) = \frac{dz}{dx}, \quad \frac{d\mathrm{F}}{dy} = \frac{dz}{dy}, \quad \frac{d\mathrm{F}}{dz} = -1,\\
&\frac{d^2\mathrm{F}}{dx^2} = \frac{d^2z}{dx^2}, \quad \frac{d^2\mathrm{F}}{dy^2} = \frac{d^2z}{dy^2}, \quad \frac{d^2\mathrm{F}}{dz^2} = 0,\\
&\frac{d^2\mathrm{F}}{dy\,dz} = 0, \quad \frac{d^2\mathrm{F}}{dz\,dx} = 0, \quad \frac{d^2\mathrm{F}}{dx\,dy} = \frac{d^2z}{dx\,dy},
\end{aligned}\right\} \;\ldots\ldots (9),$$

If we substitute these values in (8) and adopt the usual notation, we obtain the known expression

$$\rho = \mp \frac{\sqrt{1 + p^2 + q^2}}{l^2 r + 2\, lm\, s + m^2 t}.$$

5. To find the sections whose radius of curvature is a *maximum* or *minimum*, and the values of those radii.

In formula (8) let $\mp \sqrt{U^2 + V^2 + W^2} = P$, then

$$\frac{P}{\rho} = l^2 u + m^2 v + n^2 w + 2mnu' + 2nlv' + 2lmw' \ldots\ldots (10),$$

the quantities l, m, n being connected by the two equations

$$l^2 + m^2 + n^2 = 1 \ldots\ldots\ldots\ldots (11)$$

$$lU + mV + nW = 0 \ldots\ldots\ldots\ldots (12).$$

Differentiating these equations,

$$(lu + mw' + nv')\, dl + (lw' + mv + nu')\, dm$$
$$+ (lv' + mu' + nw)\, dn = 0 \ldots\ldots\ldots (13),$$

$$ldl + mdm + ndn = 0 \ldots\ldots\ldots (14),$$

$$Udl + Vdm + Wdn = 0 \ldots\ldots\ldots (15),$$

(13) + (14) $\times \lambda$ + (15) μ gives, on equating to zero, the coefficients of each differential,

$$\left.\begin{aligned} lu + mw' + nv' + \lambda l + \mu U &= 0 \\ lw' + mv + nu' + \lambda m + \mu V &= 0 \\ lv' + mu' + nw + \lambda n + \mu W &= 0 \end{aligned}\right\} \ldots\ldots (16).$$

Multiplying the first of these by l, the second by m, and the third by n, adding, and reducing by (10), (11), (12),

$$\frac{P}{\rho} + \lambda = 0.$$

Substituting for λ in (16), they become

$$\left.\begin{aligned} \left(u - \frac{P}{\rho}\right) l + w'm + v'n + \mu U &= 0 \\ w'l + \left(v - \frac{P}{\rho}\right) m + u'n + \mu V &= 0 \\ v'l + u'm + \left(w - \frac{P}{\rho}\right) n + \mu W &= 0 \end{aligned}\right\} \ldots (17).$$

To obtain the equation for ρ, it remains to eliminate l, m, n, μ from equations (17) and (12). The result is

$$U^2\left(v-\frac{P}{\rho}\right)\left(w-\frac{P}{\rho}\right)+V^2\left(w-\frac{P}{\rho}\right)\left(u-\frac{P}{\rho}\right)$$
$$+W^2\left(u-\frac{P}{\rho}\right)\left(v-\frac{P}{\rho}\right)$$
$$-2VWu'\left(u-\frac{P}{\rho}\right)-2WUv'\left(v-\frac{P}{\rho}\right)-2UVw'\left(w-\frac{P}{\rho}\right)$$
$$-U^2u'^2-V^2v'^2-W^2w'^2$$
$$+2VWv'w'+2WUw'u'+2UVu'v'$$
$$=0\ldots\ldots\ldots\ldots(18).$$

6. If the equation to the surface be of the form

$$\phi(x)+\chi(y)+\psi(z)=0,$$

which includes, among others, all the surfaces of the second order when referred to their principal diametral planes; U, V, W are functions of x, y, z alone, respectively, so that u', v', w' are all zero, and the equation (18) simplifies considerably. In short, in this case, equations (17) give immediately

$$l=\frac{\mu U}{u-\dfrac{P}{\rho}},\quad m=\frac{\mu V}{v-\dfrac{P}{\rho}},\quad n=\frac{\mu W}{w-\dfrac{P}{\rho}};$$

and by substituting these values in (12), we have

$$\frac{U^2}{u-\dfrac{P}{\rho}}+\frac{V^2}{v-\dfrac{P}{\rho}}+\frac{W^2}{w-\dfrac{P}{\rho}}=0\ldots\ldots\ldots(19).$$

7. Example. To find the principal radii of curvature of an ellipsoid.

In this case

$$F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0,$$

therefore

$$U=\frac{2x}{a^2},\quad V=\frac{2y}{b^2},\quad W=\frac{2z}{c^2},\quad u=\frac{2}{a^2},\quad v=\frac{2}{b^2},\quad w=\frac{2}{c^2},$$

and the value of P is

$$2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}=\frac{2}{p},$$

if p denote the perpendicular from the centre on the tangent plane. Substituting these values in (19), we have

$$\frac{x^2}{a^2(p\rho-a^2)}+\frac{y^2}{b^2(p\rho-b^2)}+\frac{z^2}{c^2(p\rho-c^2)}=0,$$

which may easily be put in the form

$$\rho^2-\{a^2+b^2+c^2-(x^2+y^2+z^2)\}\frac{\rho}{p}+\frac{a^2b^2c^2}{p^4}=0.$$

8. As another example, we propose to find the principal radii of curvature at any point of the surface whose equation is

$$a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2 \;\ldots\ldots\; (a),$$

and which is the locus of the extremity of the perpendicular from the centre on the tangent plane to an ellipsoid. Put

$$x^2 + y^2 + z^2 = r^2,$$

$$\text{then } F(x, y, z) = a^2x^2 + b^2y^2 + c^2z^2 - r^4,$$

$$U = 2x(a^2 - 2r^2), \quad V = 2y(b^2 - 2r^2), \quad W = 2z(c^2 - 2r^2),$$

$$u = 2(a^2 - 2r^2 - 4x^2), \quad v = 2(b^2 - 2r^2 - 4y^2), \quad w = 2(c^2 - 2r^2 - 4z^2),$$

$$u' = -8yz, \quad v' = -8zx, \quad w' = -8xy.$$

Instead of substituting these values in equation (18), we shall employ the equations from which that is deduced, namely (12) and (17), because by this method it is more easy to obtain a result of a simple form. Of these, (12) becomes

$$lx(a^2 - 2r^2) + my(b^2 - 2r^2) + nz(c^2 - 2r^2) = 0 \;\ldots\ldots\; (b),$$

and the first of (17),

$$l\left(a^2 - 2r^2 - 4x^2 - \frac{P}{\rho}\right) - 4mxy - 4nxz + \mu x(a^2 - 2r^2) = 0,$$

$$\text{or } l\left(a^2 - 2r^2 - \frac{P}{\rho}\right) - 4x(lx + my + nz) + \mu x(a^2 - 2r^2) = 0.$$

Assume $\quad lx + my + nz = k \;\ldots\ldots\ldots\ldots\; (c);$

then we have

$$\left.\begin{aligned} l\left(a^2 - 2r^2 - \frac{P}{\rho}\right) - 4kx + \mu x(a^2 - 2r^2) &= 0 \\ m\left(b^2 - 2r^2 - \frac{P}{\rho}\right) - 4ky + \mu y(b^2 - 2r^2) &= 0 \\ n\left(c^2 - 2r^2 - \frac{P}{\rho}\right) - 4kz + \mu z(c^2 - 2r^2) &= 0 \end{aligned}\right\} \;\ldots\ldots\ldots\; (d),$$

and we shall obtain the equation for ρ by eliminating l, m, n, k, μ, from the five equations (b), (c), (d). For this purpose, multiply equations (d) by x, y, z respectively, and add them, reducing by (a), (b), (c), then we find

$$k\left(\frac{P}{\rho} + 4r^2\right) + \mu r^4 = 0.$$

By this relation, eliminate μ from equations (d), and we have

$$l = \frac{4(a^2 - r^2)r^2 + (a^2 - 2r^2)\dfrac{P}{\rho}}{a^2 - 2r^2 - \dfrac{P}{\rho}} \cdot \frac{xk}{r^4},$$

with corresponding expressions for m and n. Substituting them in (b), and denoting

$$F(a, x) + F(b, y) + F(c, z) \text{ by } \Sigma F(a, x),$$

$$\Sigma \frac{4(a^2 - r^2)(a^2 - 2r^2)r^2 + (a^2 - 2r^2)^2 \dfrac{P}{\rho}}{a^2 - 2r^2 - \dfrac{P}{\rho}} \cdot x^2 = 0;$$

but by (a) $\qquad \Sigma (a^2 - r^2)\, x^2 = 0:$

multiplying the last equation by $4r^2$, subtracting from the preceding, and dividing the result by $\dfrac{P}{\rho}$,

$$\Sigma \frac{a^4 x^2}{a^2 - 2r^2 - \dfrac{P}{\rho}} = 0,$$

which is the simplest form in which the equation can be presented. The value of P will be found to be

$$2\sqrt{a^4x^2 + b^4y^2 + c^4z^2}.$$

The length of the perpendicular from the centre on the tangent plane to this surface is

$$\frac{r^4}{\sqrt{a^4x^2 + b^4y^2 + c^4z^2}},$$

therefore if p denote that line, $P = \dfrac{2r^4}{p}$.

9. The directions of the sections of greatest and least curvature will be found by eliminating λ and μ from equations (16). The result is

$$(lu + mw' + nv')(mW - nV) + (lw' + mv + nu')(nU - lW)$$
$$+ (lv' + mu' + nw')(lV - mU) = 0,$$

or $\qquad (Vv' - Ww')l^2 + (Ww' - Uu')m^2 + (Uu' - Vv')n^2$
$$+ \{Wv' - Vw' + U(v - w)\}mn + \{Uw' - Wu' + V(w - u)\}nl$$
$$+ \{Vu' - Uv' + W(u - v)\}lm = 0 \;\ldots\ldots (20),$$

which equation, together with (11) and (12), determines the values of l, m, n. It is easy to see that there will be two sets of values of l^2, m^2, n^2, and no more.

10. To prove that the directions of greatest and least curvature are at right angles.

For abbreviation, write the equation (20)

$$Ll^2 + Mm^2 + Nn^2 + L'mn + M'nl + N'lm = 0.$$

Substituting the value of n derived from equation (12),

$$(Ll^2 + Mm^2 + N'lm)W^2 - (L'm + M'l)(lU + mV)W$$
$$+ N(lU + mV)^2 = 0,$$

or $\;(MW^2 + NV^2 - L'VW)m^2 + \{2NUV - W(L'U + M'V)\}lm$
$$+ (NU^2 + LW^2 - M'WU)l^2 = 0,$$

a quadratic equation in $\frac{m}{l}$; whence we find, that if l_1, l_2 are the two values of l, and m_1, m_2 those of m,

$$\frac{m_1 m_2}{l_1 l_2} = \frac{NU^2 + LW^2 - M'WU}{MW^2 + NV^2 - L'VW};$$

so that if we assume

$$l_1 l_2 = K(MW^2 + NV^2 - L'VW),$$

we shall have $m_1 m_2 = K(NU^2 + LW^2 - M'WU)$,

and also $\quad n_1 n_2 = K(LV^2 + MU^2 - N'UV)$.

Hence, the cosine of the angle between the two directions, or

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = K\{L(V^2 + W^2) + M(W^2 + U^2) + N(U^2 + V^2) - L'VW - M'WU - N'UV\}.$$

The value of the quantity within the brackets is zero, as may easily be seen by recurring to equation (20) for the values of LM, &c, and collecting the terms multiplied by u', v', &c. respectively. Therefore the two directions are at right angles.

11. To find the lines of curvature of a given surface, or the direction in which we must pass from one point to a consecutive, in order that the normals at the two points may meet.

Retaining the same notation, the equations to the normal at a point x, y, z are

$$\frac{x' - x}{U} = \frac{y' - y}{V} = \frac{z' - z}{W}.$$

Let each member be assumed equal to Q, then

$$x' = x + QU, \quad y' = y + QV, \quad z' = z + QW.$$

If x', y', z' belong to the point in which two consecutive normals meet, the differentials of these three equations must be verified at the same time with them, therefore

$$\left.\begin{aligned} dx + QdU + UdQ &= 0, \\ dy + QdV + VdQ &= 0, \\ dz + QdW + WdQ &= 0, \end{aligned}\right\} \ldots\ldots\ldots\ldots (21),$$

or

$$u dx + w' dy + v' dz + \frac{1}{Q} dx + U \frac{dQ}{Q} = 0,$$

$$w' dx + v dy + u' dz + \frac{1}{Q} dy + V \frac{dQ}{Q} = 0,$$

$$v' dx + u' dy + w dz + \frac{1}{Q} dz + W \frac{dQ}{Q} = 0.$$

These equations are of the same form as equations (16), therefore the elimination of Q and dQ from these must give the same relation between dx, dy, dz, as the elimination of λ and μ from (16) gave between l, m, n, namely (20). And if

$$dx^2 + dy^2 + dz^2 = ds^2,$$

$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ will be the cosines of the angles which an element of a line of curvature makes with the axes, and they satisfy the same system of equations as l, m, n, namely (11), (12), (20). Therefore the directions in which two consecutive normals meet coincide with those of greatest and least curvature.

For the sake of shortness, we shall eliminate Q and dQ from equations (21), instead of their expanded forms. This is easily done by cross-multiplication, and the result is

$$\mathrm{U}(d\mathrm{V}dz - d\mathrm{W}dy) + \mathrm{V}(d\mathrm{W}dx - d\mathrm{U}dz)$$
$$+ \mathrm{W}(d\mathrm{U}dy - d\mathrm{V}dx) = 0 \ldots\ldots\ldots (22):$$

from this equation and that to the surface, the differential to the projection of a line of curvature on any coordinate plane may be found.

12. Example. To find the lines of curvature of an ellipsoid. Let the equation be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

$$\text{then } \mathrm{U} = \frac{2x}{a^2}, \quad \mathrm{V} = \frac{2y}{b^2}, \quad \mathrm{W} = \frac{2z}{c^2},$$

$$d\mathrm{U} = \frac{2}{a^2}dx, \quad d\mathrm{V} = \frac{2}{b^2}dy, \quad d\mathrm{W} = \frac{2}{c^2}dz.$$

Substituting in (22),

$$(b^2 - c^2)\,x\,dy\,dz + (c^2 - a^2)\,y\,dz\,dx + (a^2 - b^2)\,z\,dx\,dy = 0.$$

Multiply by $\frac{z}{c^2}$, and substitute the values

$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad \frac{z\,dz}{c^2} = -\frac{x\,dx}{a^2} - \frac{y\,dy}{b^2},$$

therefore

$$\{(b^2 - c^2)\,x\,dy + (c^2 - a^2)\,y\,dx\}\left(\frac{x\,dx}{a^2} + \frac{y\,dy}{b^2}\right)$$
$$-(a^2 - b^2)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)dx\,dy = 0,$$

or, reducing,

$$\frac{b^2 - c^2}{b^2}\,xy\,dy^2$$
$$+\left\{\frac{a^2 - c^2}{a^2}x^2 - \frac{b^2 - c^2}{b^2}y^2 - (a^2 - b^2)\right\}dx\,dy - \frac{a^2 - c^2}{a^2}\,xy\,dx^2 = 0.$$

To integrate this, multiply by $4xy$, and let $x^2 = u$, $y^2 = v$, then

$$\frac{b^2 - c^2}{b^2}\,u\,dv^2$$
$$+\left\{\frac{a^2 - c^2}{a^2}u - \frac{b^2 - c^2}{b^2}v - (a^2 - b^2)\right\}du\,dv - \frac{a^2 - c^2}{a^2}\,v\,du^2 = 0,$$

$$\text{or } \left(\frac{b^2-c^2}{b^2}\,dv+\frac{a^2-c^2}{a^2}\,du\right)(u\,dv-v\,du)-(a^2-b^2)\,du\,dv=0,$$

$$\text{or } u-v\frac{du}{dv}=\frac{(a^2-b^2)\dfrac{du}{dv}}{\dfrac{b^2-c^2}{b^2}+\dfrac{a^2-c^2}{a^2}\dfrac{du}{dv}},$$

an equation of Clairaut's form, whose solution is

$$u - \mathrm{C}v = \frac{\mathrm{C}\,(a^2-b^2)}{\dfrac{b^2-c^2}{b^2}+\mathrm{C}\,\dfrac{a^2-c^2}{a^2}},$$

$$\text{or } x^2 - \mathrm{C}y^2 = \frac{\mathrm{C}\,(a^2-b^2)}{\dfrac{b^2-c^2}{b^2}+\mathrm{C}\,\dfrac{a^2-c^2}{a^2}},$$

the equation to an ellipse or hyperbola. For the method of determining the constant, see *Leroy*, No. 422.

ff.

XII.—MATHEMATICAL NOTES.

1. *Taylor's Theorem.* The following proof of Taylor's Theorem by means of the separation of symbols of operation from those of quantity, may be interesting to some of our readers. Let there be an operation, which we may for convenience call D, which performed h times on $f(x)$ converts it into $f(x+h)$; that is, let $\mathrm{D}^h f(x) = f(x+h)$, we wish to determine the nature of D. Now we know that $\dfrac{d}{dx}f(x) = \dfrac{f(x+h)-f(x)}{h}$ when h is indefinitely small, therefore

$$\frac{d}{dx}f(x)=\frac{\mathrm{D}^h f(x)-f(x)}{h}=\frac{\mathrm{D}^h-1}{h}f(x)$$

when h is indefinitely small. Therefore, taking the symbols of operation separate,

$$\frac{d}{dx}=\frac{\mathrm{D}^h-1}{h} \text{ when } h=0,$$

$$\text{which gives } \mathrm{D}=\left(1+h\frac{d}{dx}\right)^{\frac{1}{h}} \text{ when } h=0.$$

Expanding by the Binomial Theorem, and making $h = 0$, we find

$$\mathrm{D}=1+\frac{d}{dx}+\frac{1}{1.2}\frac{d^2}{dx^2}+\frac{1}{1.2.3}\frac{d^3}{dx^3}+\text{\&c.}$$

$$\text{or } D = \varepsilon^{\frac{d}{dx}};$$

$$\text{therefore } f(x+h) = \varepsilon^{h\frac{d}{dx}} f(x),$$

which is Taylor's Theorem.

2. *Problem from the Papers of* 1835. If p, r be the perpendicular from the origin on the tangent plane and the radius vector of any surface, then $\frac{p^2}{r}$ will be the perpendicular on the tangent plane at the corresponding point of the surface, which is the locus of the extremity of p.

Let x, y, z be the coordinates of the first surface, α, β, γ of the second. As p is the perpendicular on the tangent plane, the equation to the plane is

$$\alpha x + \beta y + \gamma z = p^2 = \alpha^2 + \beta^2 + \gamma^2 \ldots\ldots (1).$$

Since this plane is a tangent to the surface, this equation will hold good if for x, y, z we put $x + dx$, $y + dy$, $z + dz$. Whence

$$\alpha\, dx + \beta\, dy + \gamma\, dz = 0 \ldots\ldots (2).$$

Now if $V = 0$ be the equation to the surface of which α, β, γ are the coordinates, and P be the perpendicular on the tangent plane; then

$$P = \frac{\alpha \frac{dV}{d\alpha} + \beta \frac{dV}{d\beta} + \gamma \frac{dV}{d\gamma}}{\left\{\left(\frac{dV}{d\alpha}\right)^2 + \left(\frac{dV}{d\beta}\right)^2 + \left(\frac{dV}{d\gamma}\right)^2\right\}^{\frac{1}{2}}};$$

$$\text{also, } \frac{dV}{d\alpha} d\alpha + \frac{dV}{d\beta} d\beta + \frac{dV}{d\gamma} d\gamma = 0 \ldots\ldots\ldots\ldots (3).$$

Now differentiating (1), considering $\alpha, \beta, \gamma, x, y, z$ as variables, and paying regard to (2), we have

$$(x - 2\alpha)\, d\alpha + (y - 2\beta)\, d\beta + (z - 2\gamma)\, d\gamma = 0 \ldots\ldots (4),$$

$\lambda\,(3)-(4)$ gives, on equating to 0, the coefficients of each of the differentials

$$\lambda \frac{dV}{d\alpha} = x - 2\alpha, \quad \lambda \frac{dV}{d\beta} = y - 2\beta, \quad \lambda \frac{dV}{d\gamma} = z - 2\gamma.$$

Substituting these in the expression for P, it becomes

$$P = \frac{2(\alpha^2 + \beta^2 + \gamma^2) - (\alpha x + \beta y + \gamma z)}{\{x^2 + y^2 + z^2 + 4[\alpha^2 + \beta^2 + \gamma^2 - (\alpha x + \beta y + \gamma z)]\}^{\frac{1}{2}}},$$

which by (1) is reduced to

$$P = \frac{\alpha^2 + \beta^2 + \gamma^2}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{p^2}{r}.$$

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